

# EXPLICIT CONSTRUCTION OF A ROBUST FAMILY OF COMPACT INERTIAL MANIFOLDS

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**ABSTRACT.** A construction of a robust family of compact inertial manifolds is presented. The result aims to complete an analysis of certain types of attracting sets for a class of dissipative infinite dimensional dynamical systems. Application to a hyperbolically relaxed Chafee-Infante reaction diffusion equation is also discussed.

## 1. INTRODUCTION

The aim of this article is to construct a robust family (both upper- and lower-semicontinuous with explicit control over semidistances) of compact inertial manifolds for a one-parameter family of semiflows, and to study the possible implementation of this construction for the semiflows generated by a dissipative nonlinear evolution equation depending on a parameter. In the category of dissipative infinite dimensional dynamical systems, it is well-known that, under suitable assumptions, semiflows admit an inertial manifold. The continuity properties of these manifolds with respect to perturbations of the differential equation, and hence the corresponding semiflow, is an open problem. As an intermediate step in establishing a result as to the existence (or non-existence) of a robust family of inertial manifolds, we will examine a robust family of *compact* inertial submanifolds. There is already in the literature a generic result on the existence of compact inertial submanifolds. An improvement of this result will be given by constructing a compact inertial manifold satisfying a certain regularity property, a property that will be exploited when we explore the continuity properties of these sets under perturbations. The outcome from these two results will be a construction of a robust family of compact inertial manifolds.

The motivation for the development of this article is the work of [13] in (indirect) response to [24, Rem. 4.13]:

As was the case for the global attractors, a natural question is to know whether any result on the upper- or lower-semicontinuity would hold for the exponential attractors[.]

As per another remark from [24, §5.8.6, ¶1]:

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It would in fact be interesting to further investigate this question for example in the framework of the upper- and lower-semicontinuity of these inertial manifolds as [the perturbation parameter vanishes].

Indeed, attempts to resolve this natural question appear in the papers [5] and [6]. Although, the presentation therein is technically difficult since it lacks a complete reworking of the functional setting. The idea in each is to extend the framework for the hyperbolic perturbation problem so as to include the framework of the limiting case parabolic problem. In [5], a construction of exponential attractors which satisfy both upper- and lower-semicontinuity properties in this extended framework is given. A construction of a family of exponential attractors satisfying a further, robust, type of continuity, again in an extended framework, is given. The difficulty of this approach comes from the fact that this “extended framework” is not natural to either problem’s traditional setting.

To overcome this, a new approach was taken whereby one *lifts* only the exponential attractor from the parabolic case into the natural product setting of the hyperbolic problem. In fact, this is exactly how one proceeds to show the upper-semicontinuity result in [24] for the global attractors of the model problem. However, the procedure requires an extra degree of regularity from the parabolic attractors. Whereas the global attractors are regular enough to be lifted, the exponential attractors are not. Borrowing some of the main ideas in [6], [13] successively established this new approach. Based on this idea, a construction of a regular *robust* family of exponential attractors was attainable without encountering the difficulties found in the previous attempts.

Results concerning the existence of robust families of attracting sets appear in the literature for equations depending on parameters that occur as a hyperbolic relaxation, in a so-called memory kernel, and in problems with dynamic boundary conditions; for example, domains with permeable walls. It was from [5], [6], [25], and ultimately [13]<sup>1</sup>, that robustness matured and led to the many results found in [14], [15], and [18] to name three models with a singular perturbation of hyperbolic relaxation type. Furthermore, [16] and [3] contain results where the perturbation parameter lies in a memory kernel, and the following [26], [9], [7], [8], [10], [11], and [12] involve dynamic boundary conditions. (Many other results are certainly coming.) With respect to the many models described in the sources just cited, the model we will later investigate, as motivated by the application we give, concerns a singular perturbation of hyperbolic relaxation type.

As far as we know, a construction of a robust family of inertial manifolds does not yet appear in the literature. Based on the current methods, in order to show that a family of inertial manifolds—in their totality—is robust, one would need to establish that the manifolds satisfy a regularity property which is somewhat unnatural for such attractors. On the other hand, the “compact core” of an inertial manifold is regular enough to construct a family of robust compact inertial manifolds. To determine the applicability of the construction given in this article, the existence of a robust family of compact inertial manifolds will be demonstrated for a perturbed dissipative evolution equation of hyperbolic relaxation type.

In section 2, the model initial boundary value problem is presented. Here we recall the fact that the mild solutions generate a Lipschitz continuous semiflow. We

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<sup>1</sup>Marco Squassina’s lecture, “Singular Limit of Differential Systems with Memory,” available at <http://www.dmf.unicatt.it/squassina/papers/lavori/pomona04.pdf>, is also interesting.

also recall assumptions on the equation which provide the existence of an inertial manifold of graph type. The existence of compact inertial manifolds is discussed in section 3, which begins with a generic construction found in [24], and then leads into an explicit construction of a compact inertial manifold which is well suited for the robustness result shown in section 4. The product structure where the robustness takes place, as well as the necessary maps used to “lift” certain sets in this structure, is given in section 4. It should be mentioned that the robustness result for the compact inertial manifolds will make use of a modified version of an assumption described in the seminal work [13]. In section 5 we investigate the one-dimensional Chafee-Infante reaction diffusion equation with a singular perturbation of hyperbolic relaxation type. It is already known that the solution operators for the equation admit an upper-continuous family of global attractors (cf. [19]) and a robust family of exponential attractors (cf. [13]). It is also well-known that the solution operators admit a family of inertial manifolds, however nothing is known about the stability of this family. We will show that the solution operators admit a family of compact inertial manifolds—thus illuminating the previous section—which possess a *weaker* version of the robustness property; that is, weaker in the sense that the semidistances go to zero provided that the dimension of the manifolds is allowed to increase without bound. We only treat the one-dimensional case here because, quite simply, it is the easiest case to work with. There is a possibility of constructing families of compact inertial manifolds for partial differential equations whose unknown depends on a spatial variable in two or three dimensions (four or more is unlikely; on these comments, see [22]).

## 2. PRELIMINARIES

Consider the initial value problem,

$$(1) \quad \begin{cases} \frac{du(t)}{dt} + Au(t) = F(u(t)), & t > 0 \\ u(0) = u_0, \end{cases}$$

where  $-A$  is the infinitesimal generator of a strongly continuous ( $C^0$ -)semigroup  $e^{-At}$ ,  $t \geq 0$ , on a real Hilbert space  $\mathcal{X}$ , and  $F : \mathcal{X} \rightarrow \mathcal{X}$  is globally Lipschitz continuous. For every  $u_0 \in \mathcal{X}$  and  $T > 0$ , problem (1) has a unique mild solution  $u \in C([0, T]; \mathcal{X})$  satisfying the integral equation, for all  $t \in [0, T]$ ,

$$(2) \quad u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-\tau)}F(u(\tau))d\tau.$$

Moreover, for all  $T > 0$  the map

$$\mathcal{X} \ni u_0 \mapsto u \in C([0, T]; \mathcal{X})$$

is Lipschitz continuous (cf. [27, Chap. 6, Thm. 1.2]), and thus the mild solutions of problem (1) generate a Lipschitz continuous *semiflow*  $S = (S(t))_{t \geq 0}$  in  $\mathcal{X}$  where

$$S(t)u_0 := u(t, u_0).$$

Precisely, the maps  $S(t) : \mathcal{X} \rightarrow \mathcal{X}$  satisfy the following:

- (i) For each fixed  $u_0 \in \mathcal{X}$ , the map  $t \mapsto S(t)u_0$  is continuous.
- (ii) For each fixed  $t \geq 0$ , the map  $u_0 \mapsto S(t)u_0$  is Lipschitz continuous.
- (iii)  $S(0) = I_{\mathcal{X}}$  (identity in  $\mathcal{X}$ ).
- (iv)  $S(s+t) = S(s)S(t)$  for all  $s, t \geq 0$ .

Properties (i) and (ii) follow directly from [27], and (iii) is a simple consequence of the representation of the mild solution (2). Property (iv) is known to hold for a general class of autonomous IVP including the one under consideration here (cf. [24, §1.2.4]).

Let  $\Sigma_1$  be an  $N$ -dimensional closed linear subspace of  $\mathcal{X}$ . By the projection theorem,  $\mathcal{X}$  can be decomposed uniquely into the direct sum of  $\Sigma_1$  and its orthogonal complement

$$\Sigma_2 := \Sigma_1^\perp = \{x \in \mathcal{X} : \langle x, y \rangle_{\mathcal{X}} = 0, \forall y \in \Sigma_1\};$$

i.e.  $\mathcal{X} = \Sigma_1 \oplus \Sigma_2$ ; hence, every  $x \in \mathcal{X}$  can be decomposed into the sum  $x = p + q$  for some  $p \in \Sigma_1$  and  $q \in \Sigma_2$ . The continuous maps

$$P_N : \mathcal{X} \rightarrow \Sigma_1 \text{ and } Q_N := I_{\mathcal{X}} - P_N : \mathcal{X} \rightarrow \Sigma_2$$

given by

$$P_N x = p \text{ and } Q_N x = q,$$

are *orthogonal projections* of  $\mathcal{X}$  onto  $\Sigma_1$  and  $\Sigma_2$  respectively.

A subset  $\mathcal{M} \subseteq \mathcal{X}$  is called an  $N$ -dimensional Lipschitz submanifold of  $\mathcal{X}$  if it satisfies the following:

- (1) There exists a countable (possibly finite) collection of open sets  $V_i \subseteq \mathcal{X}$ ,  $i \in \Lambda \subseteq \mathbb{N}$ , such that if  $U_i := V_i \cap \mathcal{M}$ , then  $\mathcal{M} = \cup_{i \in \Lambda} U_i$ .
- (2) There exist open sets  $W_i \subseteq \mathbb{R}^N$  and invertible mappings  $\phi_i : W_i \rightarrow V_i$ , with  $\phi_i(W_i \cap \mathbb{H}^N) = U_i$ , such that both  $\phi_i$  and  $\phi_i^{-1}$  are Lipschitz continuous. Here,

$$\mathbb{H}^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 \geq 0\}.$$

Denote by

$$\partial\mathcal{M} := \bigcup_{i \in \Lambda} \phi_i(W_i \cap \partial\mathbb{H}^N)$$

the *boundary* of  $\mathcal{M}$ .

A subset  $\mathcal{M} \subseteq \mathcal{X}$  is called an  $N$ -dimensional trivial Lipschitz submanifold of  $\mathcal{X}$  if it satisfies the following:

- (i) There exists a closed  $N$ -dimensional linear subspace  $\Sigma_1$  of  $\mathcal{X}$ , and a subset  $\Gamma$  of  $\Sigma_1$  such that  $\Gamma$  is an  $N$ -dimensional Lipschitz manifold.
- (ii) There exists an invertible mapping  $\varphi : \Gamma \rightarrow \mathcal{M}$  such that both  $\varphi$  and  $\varphi^{-1}$  are Lipschitz continuous.

A subset  $\mathcal{M} \subseteq \mathcal{X}$  is an *inertial manifold* for the semiflow  $S$  if  $\mathcal{M}$  is an  $N$ -dimensional trivial Lipschitz submanifold of  $\mathcal{X}$  which is *positively invariant* and *exponentially attracting* for  $S$ ; i.e., for all  $t \geq 0$ ,

$$S(t)\mathcal{M} \subseteq \mathcal{M},$$

and for any bounded subset  $B$  of  $\mathcal{X}$ , there exist  $c \geq 0$  and  $\omega > 0$ , depending on  $B$ , such that for all  $t \geq 0$ ,

$$\partial_{\mathcal{X}}(S(t)B, \mathcal{M}) \leq ce^{-\omega t}.$$

Here,

$$\partial_{\mathcal{X}}(U, V) := \sup_{x \in U} \inf_{y \in V} \|x - y\|_{\mathcal{X}},$$

denotes the *Hausdorff semidistance*, in the topology of  $\mathcal{X}$ , between the two subsets  $U$  and  $V$  of  $\mathcal{X}$ .

We now state the first hypothesis of the article.

*Hypothesis 2.1.* In equation (1), assume  $F \in C_b(\mathcal{X}, \mathcal{X})$  satisfies the global Lipschitz condition

$$(3) \quad \|F(x) - F(y)\|_{\mathcal{X}} \leq \ell_F \|x - y\|_{\mathcal{X}}$$

for some  $\ell_F > 0$ , independent of  $x$  and  $y$ , but depending on  $F$ .

Define the subset  $\mathcal{G}_L$  of  $\mathcal{G} := C_b(\Sigma_1; \Sigma_2)$  by

$$\mathcal{G}_L := \{\Psi \in \mathcal{G} : \|\Psi(\chi) - \Psi(\psi)\|_{\mathcal{X}} \leq L\|\chi - \psi\|_{\mathcal{X}}, \forall \chi, \psi \in \Sigma_1\}.$$

In some methods of constructing inertial manifolds (cf. e.g. [31, Chap. VIII], [29, §15.2], and [24, §5.2.3], and the references therein), the inertial manifold  $\mathcal{M}$  is constructed as the graph of a Lipschitz continuous function  $m$  defined on a closed subset  $\Gamma$  of a finite-dimensional closed linear subspace  $\Sigma_1$  of the Hilbert space  $\mathcal{X}$ . Indeed, for the remainder of this article, any inertial manifold  $\mathcal{M}$  under consideration is assumed to be constructed in such a fashion because the applications known to admit an inertial manifold, admit one of *graph type*. Precisely, for some  $N$ -dimensional closed linear subspace  $\Sigma_1$  of  $\mathcal{X}$ , with an  $N$ -dimensional Lipschitz submanifold  $\Gamma \subseteq \Sigma_1$ , and for some  $m \in \mathcal{G}_{\ell_F}$ , an *inertial manifold of graph type* is described by

$$(4) \quad \mathcal{M} := \text{graph}(m) = \{\xi + m(\xi) : \xi \in \Gamma\}.$$

Such a set is an  $N$ -dimensional trivial Lipschitz submanifold in accordance with the definition above. Additionally, the manifold  $\mathcal{M}$  is closed. The function  $m \in \mathcal{G}_{\ell_F}$  above is obtained through Hadamard's graph transformation method. This procedure provides the existence of an inertial manifold for a semiflow satisfying a so-called strong squeezing property. We will consider here semiflows which satisfy a stronger condition whereby the operator  $A$  in (1) satisfies a spectral gap condition. This also allows us to determine the appropriate dimension  $N \geq 1$  of the closed linear subspace  $\Sigma_1$  of  $\mathcal{X}$ .

The operator  $A$  is said to satisfy the *spectral gap condition relative to  $F$*  provided there is  $N \in \mathbb{N}$  such that the real parts of the eigenvalues of  $A$  satisfy

$$(5) \quad \Re(\lambda_{N+1}) - \Re(\lambda_N) > 4\ell_F,$$

with  $N$  so large that

$$(6) \quad \Re(\lambda_{N+1}) > 2\ell_F.$$

When  $F$  satisfies hypothesis 2.1 and  $A$  satisfies the spectral gap condition, then it is well-known that the resulting semiflow  $S$  admits an inertial manifold  $\mathcal{M}$  of graph type in  $\mathcal{X}$  (cf. [24, Thm. 5.42]); i.e., there is an  $N$ -dimensional closed linear subspace  $\Sigma_1$  of  $\mathcal{X}$  (where  $N \geq 1$  was determined by (5)-(6) above), and there is  $m \in \mathcal{G}_{\ell_F}$  (where  $\ell_F > 0$  was determined in (3)) such that the set  $\mathcal{M} := \text{graph}(m)$  is an inertial manifold for the semiflow  $S$  in  $\mathcal{X}$ . In particular, when  $A^{-1}$  is compact on  $\mathcal{X}$ , the set of eigenvectors of  $A$ , say  $\{\omega_k\}_{k=1}^{\infty}$ , forms a complete orthonormal system in  $\mathcal{X}$ . With this, the finite-dimensional closed linear subspace  $\Sigma_1$  may be described by  $\Sigma_1 = \text{span}\{\omega_k\}_{k=1}^N$ . Also,  $\mathcal{M}$  possesses the following *exponential tracking property*: there exists  $c, \eta > 0$  such that for all  $x \in \mathcal{X}$ , there is  $x' \in \mathcal{M}$ , depending on  $x$ , such that for all  $t \geq 0$ ,

$$(7) \quad \|S(t)x - S(t)x'\|_{\mathcal{X}} \leq c\|\pi_2 x - m(\pi_1 x)\|_{\mathcal{X}}e^{-\eta t}.$$

Observe that here, the evolution of the trajectories is estimated in terms of  $x$  alone. This exponential tracking property is a form of asymptotic completeness:

given any trajectory  $(S(t)x)_{t \geq 0}$  of the dynamical system, there is another trajectory  $(S(t)x')_{t \geq 0}$  on  $\mathcal{M}$ , which must then stay on  $\mathcal{M}$ , to which, not only does  $S(t)x$  exponentially converge, but further, the evolution of the trajectories is estimated in terms of the distance of  $x$  to  $\mathcal{M}$ .

Before we conclude this section we summarize the assumptions made on  $A$ .

*Hypothesis 2.2.* Assume that the operator  $A$  satisfies the following:

- (1)  $-A$  generates a  $C^0$ -semigroup on  $\mathcal{X}$ .
- (2) There is a minimum  $N_* \in \mathbb{N}$  such that for each  $N \geq N_*$ ,  $A$  satisfies the spectral gap condition relative to  $F$ .

Additionally, we may also assume the following:

- 3. The spaces  $\mathcal{Y}$ ,  $\mathcal{X}$ , and  $\mathcal{Y}'$  ( $\mathcal{Y}'$  denotes the dual of  $\mathcal{Y}$ ) form a Gelfand triple; i.e.,  $\mathcal{Y}$  is a dense subspace of  $\mathcal{X}$  satisfying  $\mathcal{Y} \hookrightarrow \mathcal{X} \cong \mathcal{X}' \hookrightarrow \mathcal{Y}'$  with compact injection  $\mathcal{Y} \hookrightarrow \mathcal{X}$  (recall  $\mathcal{X}$  is a Hilbert space it is isometrically isomorphic to its dual  $\mathcal{X}'$ ).
- 4.  $A \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$  (with  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$  being not necessarily bounded) is self-adjoint and strictly positive.

Throughout the article we assume that  $S = (S(t))_{t \geq 0}$  is a continuous semiflow on the Hilbert space  $\mathcal{X}$ , and that  $S$  admits an inertial manifold  $\mathcal{M}$  of graph type in  $\mathcal{X}$ , which possesses the exponential tracking property (7).

### 3. THE EXISTENCE OF COMPACT INERTIAL MANIFOLDS

**3.1. Motivation.** A subset  $G \subset \mathcal{X}$  is *absorbing* with respect to the semiflow  $S$ , relative to a neighborhood  $U$  of  $G$  in  $\mathcal{X}$ , if for every bounded subset  $V$  of  $U$ , there exists  $t_0 \geq 0$ , depending on  $V$ , such that  $S(t)V \subseteq G$  for all  $t \geq t_0$ .

Given  $r > 0$  and a (relatively) compact subset  $K \subset \mathcal{X}$ , we denote by  $M_{\mathcal{X}}(K, r)$  the minimum number of  $r$ -balls of  $\mathcal{X}$  necessary to cover  $K$ . The *fractal dimension* of  $K$  is defined as

$$\dim_F(K) := \limsup_{r \rightarrow 0} \frac{\ln M_{\mathcal{X}}(K, r)}{\ln \frac{1}{r}}.$$

The following citation provides the motivation for this article.

*Proposition 3.1.* Let  $\mathcal{M}$  be an inertial manifold for the semiflow  $S$  in  $\mathcal{X}$ . Assume that  $\mathcal{M}$  satisfies the following exponential tracking property: there exist  $c, \eta > 0$  such that for all  $x \in \mathcal{X}$ , there is  $x' \in \mathcal{M}$ , depending on  $x$ , such that for all  $t \geq 0$ ,

$$\|S(t)x - S(t)x'\|_{\mathcal{X}} \leq c\partial_{\mathcal{X}}(x, \mathcal{M})e^{-\eta t} = c \inf_{y \in \mathcal{M}} \|x - y\|_{\mathcal{X}} e^{-\eta t}.$$

Furthermore, assume that  $S$  admits a compact absorbing positively invariant set  $G$  in  $\mathcal{X}$ . If the set  $\widetilde{\mathcal{M}} := \mathcal{M} \cap G$  is non-empty and has Lipschitz boundary, then  $\widetilde{\mathcal{M}}$  is a compact inertial manifold for  $S$  in  $\mathcal{X}$ ; precisely,

- (i)  $\widetilde{\mathcal{M}}$  is a compact set in  $\mathcal{X}$ ,
- (ii)  $\widetilde{\mathcal{M}}$  is positively invariant under the semiflow  $S$ ,
- (iii)  $\widetilde{\mathcal{M}}$  is exponentially attracting in  $\mathcal{X}$ ,  
and
- (iv)  $\dim_F(\widetilde{\mathcal{M}}) < \infty$ .

*Proof.* Cf. [24, Proposition 5.9]. □

What follows is a construction of a non-empty set  $\widehat{\mathcal{M}}$  in  $\widetilde{\mathcal{M}}$  that possesses a Lipschitz boundary and a certain degree of regularity (which will be of much use later on when we discuss robustness).

**3.2. The construction (part 1).** A family of continuous operators  $Z = (Z(t))_{t \geq 0}$  is called *uniformly decaying to zero* if for any bounded set  $B$  in  $\mathcal{X}$ ,

$$\lim_{t \rightarrow \infty} \sup_{x \in B} \|Z(t)x\|_{\mathcal{X}} = 0.$$

A family of continuous operators  $K = (K(t))_{t \geq 0}$  is called *uniformly compact for large t* if for any bounded set  $B$  in  $\mathcal{X}$  there exists a  $t_1 \geq 0$ , depending on  $B$ , such that the set

$$\bigcup_{t \geq t_1} K(t)B$$

is relatively compact in  $\mathcal{X}$ .

*Hypothesis 3.2.* Assume that the semiflow  $S = (S(t))_{t \geq 0}$  admits a decomposition such that  $S = Z + K$  where both  $Z = (Z(t))_{t \geq 0}$  and  $K = (K(t))_{t \geq 0}$  are families of continuous operators in  $\mathcal{X}$  (not necessarily semiflows), and  $Z$  and  $K$  are, respectively, uniformly decaying to zero and uniformly compact for large  $t$ .

Let  $U \subseteq \mathcal{X}$ . The set

$$\omega(U) := \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)U}^{\mathcal{X}}$$

is the  *$\omega$ -limit set of  $U$* .

*Theorem 3.3.* Assume the continuous semiflow  $S$  in  $\mathcal{X}$  satisfies hypothesis 3.2. Then for every non-empty bounded subset  $B$  of  $\mathcal{X}$ , the set  $\omega(B)$  is non-empty, compact in  $\mathcal{X}$ , invariant under  $S$ , and *attracts*  $B$ ; i.e.,

$$\lim_{t \rightarrow \infty} \partial_{\mathcal{X}}(S(t)B, \omega(B)) = 0.$$

*Proof.* Cf. [24, Prop. 2.49]. □

*Lemma 3.4.* Suppose  $B$  is a bounded subset of a Hilbert space  $\mathcal{H}$  and  $S$  is a semiflow on  $\mathcal{H}$  satisfying hypothesis 3.2. Then

$$(8) \quad \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} K(t)B}^{\mathcal{H}} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}^{\mathcal{H}}.$$

*Proof.* First assume

$$z \in \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} K(t)B}^{\mathcal{H}}.$$

Then

$$z \in \overline{\bigcup_{t \geq s} K(t)B}^{\mathcal{H}} \quad \forall s \geq 0;$$

thus, for all  $s \geq 0$ , there exist sequences

$$(z_s^n)_{n \geq 0} \subset \bigcup_{t \geq s} K(t)B$$

such that  $z_s^n \rightarrow z$  in  $\mathcal{H}$ . Now, for all  $n \geq 0$  there exists  $t_n \geq s$  and  $b_n \in B$  such that  $z_s^n = K(t_n)b_n$ . Consequently,

$$z_s^n = S(t_n)b_n - Z(t_n)b_n$$

so that

$$S(t_n)b_n = z_s^n + Z(t_n)b_n \rightarrow z.$$

This means

$$z \in \overline{\bigcup_{t \geq s} S(t)B}^{\mathcal{H}},$$

and it follows that

$$z \in \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}^{\mathcal{H}}.$$

We have thus established “ $\subseteq$ ” in (8). The other direction is analogous.  $\square$

*Hypothesis 3.5.* Assume that  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ , and  $\mathcal{X}_3$  are Hilbert spaces which satisfy the continuous and compact imbeddings into  $\mathcal{X}$ :  $\mathcal{X}_3 \hookrightarrow \mathcal{X}_2 \hookrightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}$ . Additionally, assume that  $\mathcal{X}_1 \hookrightarrow \mathcal{X}$  densely.

Let  $\Gamma \subseteq \Sigma_1$  and for a bounded subset  $\mathcal{W}$  of  $\Gamma$ , define the bounded subset of the inertial manifold  $\mathcal{M}$

$$(9) \quad \mathcal{M}_{\mathcal{W}} := \text{graph}|_{\mathcal{W}}(m) = \{\xi + m(\xi) : \xi \in \mathcal{W}\}.$$

*Hypothesis 3.6.* Set  $C_0 := \mathcal{M}_{\mathcal{W}}$ . For  $j = 0, 1, 2$ , assume that there exist  $c_j \geq 0$  such that

$$C_{j+1} := \bigcup_{t \geq c_j} K(t)C_j$$

is bounded in  $\mathcal{X}_{j+1}$ . It is important to note that the sets  $C_j$  are relatively compact in  $\mathcal{X}$ , but they are not necessarily bounded in  $\mathcal{X}_{j+1}$ . This hypothesis is a regularity requirement on the sets  $C_{j+1}$ .

*Lemma 3.7.* Assume hypotheses 3.2, 3.5, and 3.6 hold. There holds, for  $j = 0, 1, 2$ ,

$$\lim_{t \rightarrow \infty} \partial_{\mathcal{X}}(K(t)C_j, \mathcal{M}) = 0.$$

*Proof.* Case  $j = 0$ :

$$\begin{aligned} \partial_{\mathcal{X}}(K(t)C_0, \mathcal{M}) &\leq \sup_{x \in C_0} \inf_{y \in \mathcal{M}} \|K(t)x - y\|_{\mathcal{X}} \\ &\leq \sup_{x \in C_0} \inf_{y \in \mathcal{M}} \|S(t)x - y\|_{\mathcal{X}} + \sup_{x \in C_0} \|Z(t)x\|_{\mathcal{X}}. \end{aligned}$$

Since  $C_0 = \mathcal{M}_{\mathcal{W}} \subset \mathcal{M}$  and  $\mathcal{M}$  is positively invariant, then

$$\sup_{x \in C_0} \inf_{y \in \mathcal{M}} \|S(t)x - y\|_{\mathcal{X}} = 0,$$

and since  $C_0$  is bounded and  $Z$  is uniformly decaying to zero, then the claim is proved for  $j = 0$ .

Case  $j = 1, 2$ : For all  $y \in \mathcal{M}$ ,

$$(10) \quad \partial_{\mathcal{X}}(K(t)C_j, \mathcal{M}) \leq \max\left\{\sup_{x \in C_j \cap \mathcal{M}} \|K(t)x - y\|_{\mathcal{X}}, \sup_{x \in C_j \setminus \mathcal{M}} \|K(t)x - y\|_{\mathcal{X}}\right\}.$$

Let  $x \in C_j$ . If  $x \in C_j \cap \mathcal{M}$ , we can take  $y = S(t)x \in S(t)\mathcal{M} \subseteq \mathcal{M}$  (because  $\mathcal{M}$  is positively invariant) to deduce

$$(11) \quad \sup_{x \in C_j \cap \mathcal{M}} \|K(t)x - S(t)x\|_{\mathcal{X}} = \sup_{x \in C_j \cap \mathcal{M}} \|Z(t)x\|_{\mathcal{X}}.$$

If  $x \in C_j \setminus \mathcal{M}$ , we apply the exponential tracking property from (7): there is  $x' \in \mathcal{M}$ , depending on  $x$ , and there exist  $c, \eta > 0$ , independent of  $x$ , such that for all  $t \geq 0$ ,

$$\|S(t)x - S(t)x'\|_{\mathcal{X}} \leq c\|Q_Nx - m(P_Nx)\|_{\mathcal{X}}e^{-\eta t}.$$

Then, taking  $y = S(t)x' \in \mathcal{M}$ , we deduce that

$$(12) \quad \begin{aligned} \|K(t)x - S(t)x'\|_{\mathcal{X}} &\leq \|Z(t)x\|_{\mathcal{X}} + \|S(t)x - S(t)x'\|_{\mathcal{X}} \\ &\leq \|Z(t)x\|_{\mathcal{X}} + c\|Q_Nx - m(P_Nx)\|_{\mathcal{X}}e^{-\eta t}. \end{aligned}$$

Since  $C_j$  is bounded (in  $\mathcal{X}_j \hookrightarrow \mathcal{X}$ ), we conclude from (12) that for all  $x \in C_j$ ,

$$(13) \quad \sup_{x \in C_j \setminus \mathcal{M}} \|K(t)x - y\|_{\mathcal{X}} \leq \sup_{x \in C_j \setminus \mathcal{M}} \|Z(t)x\|_{\mathcal{X}} + ce^{-\eta t}$$

with  $c > 0$  independent of  $x$ . After collecting (10), (11), and (13), and using the fact that  $Z$  is uniformly decaying to zero, the conclusion follows.  $\square$

*Hypothesis 3.8.* Assume that the semiflow  $S$  is continuous in  $\mathcal{X}_{j+1}$  for  $j = 0, 1, 2$ ; that is, for all  $t \geq 0$ ,  $S(t)\mathcal{X}_{j+1} \subseteq \mathcal{X}_{j+1}$ . This is a regularity requirement on  $S$ .

Define, for  $j = 0, 1, 2$ ,

$$(14) \quad \omega_{c_j}^K(C_j) := \overline{\bigcup_{s \geq c_j} K(t)C_j}^{\mathcal{X}_{j+1}}.$$

*Lemma 3.9.* Assume hypotheses 3.2, 3.5, 3.6, and 3.8 hold. For  $j = 0, 1, 2$  and for each  $c_j \geq 0$ , the set  $\omega_{c_j}^K(C_j)$  defined in (14) is non-empty, compact in  $\mathcal{X}$ , positively invariant under  $S$ , and satisfies

$$(15) \quad \omega_{c_j}^K(C_j) \subseteq \omega_0^K(C_j) \subseteq \omega(C_j) \subseteq \mathcal{M} \cap \mathcal{X}_{j+1}.$$

Furthermore, each set  $\omega_{c_j}^K(C_j)$  possesses finite fractal dimension and a Lipschitz smooth boundary.

*Proof.* 1. To show that the sets  $\omega_{c_j}^K(C_j)$ ,  $j = 0, 1, 2$ , are non-empty, let  $x_0$  be an element of the non-empty set  $\mathcal{M}_W = C_0$ . Then define

$$x_1 := K(c_0)x_0 \in \bigcup_{t \geq c_0} K(t)C_0 = C_1, \text{ and } x_2 := K(c_1)x_1 \in \bigcup_{t \geq c_1} K(t)C_1 = C_2.$$

For  $j \in \{0, 1, 2\}$ , define  $z_j := \lim_{t \rightarrow \infty} K(t)x_j$  in  $\mathcal{X}_{j+1}$ . Hence, by hypothesis 3.6,

$$z_j \in \overline{\bigcup_{t \geq s} K(t)C_j}^{\mathcal{X}_{j+1}}$$

for all  $s \geq c_j$ ; that is, by definition (14),

$$z_j \in \overline{\bigcup_{s \geq c_j} \bigcup_{t \geq s} K(t)C_j}^{\mathcal{X}_{j+1}} = \omega_{c_j}^K(C_j).$$

2. To show positive invariance, let  $\tau \geq 0$  be given. Then

$$\begin{aligned} S(\tau)\omega_{c_j}^K(C_j) &= S(\tau) \bigcap_{s \geq c_j} \overline{\bigcup_{t \geq s} K(t)C_j}^{\mathcal{X}_{j+1}} = S(\tau) \bigcap_{s \geq c_j} \overline{\bigcup_{t \geq s} S(t)C_j}^{\mathcal{X}_{j+1}} \\ &= \bigcap_{s \geq c_j} \overline{\bigcup_{t \geq s} S(t+\tau)C_j}^{\mathcal{X}_{j+1}} = \bigcap_{s \geq c_j} \overline{\bigcup_{\theta \geq s+\tau} S(\theta)C_j}^{\mathcal{X}_{j+1}} \\ &\subseteq \bigcap_{s \geq c_j} \overline{\bigcup_{\theta \geq s} S(\theta)C_j}^{\mathcal{X}_{j+1}} = \bigcap_{s \geq c_j} \overline{\bigcup_{\theta \geq s} K(\theta)C_j}^{\mathcal{X}_{j+1}} = \omega_{c_j}^K(C_j). \end{aligned}$$

Thus  $S(\tau)\omega_{c_j}^K(C_j) \subseteq \omega_{c_j}^K(C_j)$  for all  $\tau \geq 0$ .

3. By construction,  $\omega_0^K(C_j)$  is a bounded subset of  $\mathcal{X}_{j+1}$ . The first inclusion in (15) follows from

$$\begin{aligned} \omega_{c_j}^K(C_j) &= \bigcap_{s \geq c_j} \overline{\bigcup_{t \geq s} K(t)C_j}^{\mathcal{X}_{j+1}} \subseteq \bigcap_{s-c_j \geq 0} \overline{\bigcup_{t \geq s-c_j} K(t)C_j}^{\mathcal{X}_{j+1}} \\ &= \bigcap_{r \geq 0} \overline{\bigcup_{t \geq r} K(t)C_j}^{\mathcal{X}_{j+1}} = \omega_0^K(C_j). \end{aligned}$$

For the second inclusion, we use the fact that  $\mathcal{X}_{j+1} \subset \mathcal{X}$ ,

$$\omega_0^K(C_j) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} K(t)C_j}^{\mathcal{X}_{j+1}} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)C_j}^{\mathcal{X}_{j+1}} \subseteq \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)C_j}^{\mathcal{X}} = \omega(C_j).$$

For the last inclusion we need to show that  $\omega(C_j) \subseteq \mathcal{M}$ : Since  $\omega(C_j)$  is a bounded set in  $\mathcal{X}$ , it is attracted by  $\mathcal{M}$ ; i.e.

$$\lim_{t \rightarrow \infty} \partial_{\mathcal{X}}(S(t)\omega(C_j), \mathcal{M}) = 0.$$

Since  $\omega(C_j)$  is invariant,  $\partial_{\mathcal{X}}(\omega(C_j), \mathcal{M}) = 0$ , and since  $\omega(C_j)$  is compact and  $\mathcal{M}$  is closed in  $\mathcal{X}$ , then (cf. e.g. [24, Prop. 2.2])  $\omega(C_j) \subseteq \mathcal{M}$ . Thus (15) holds.

4. The sets  $\omega_{c_j}^K(C_j)$  are compact in  $\mathcal{X}$  thanks to the compactness of the imbeddings  $\mathcal{X}_{j+1} \hookrightarrow \mathcal{X}$ .

5.  $\dim_F(\omega_{c_j}^K(C_j)) \leq \dim_F(\mathcal{M}) = N$  because  $\omega_{c_j}^K(C_j)$  is a compact subset of the  $N$ -dimensional manifold  $\mathcal{M}$ .

6. We now show that each set  $\omega_{c_j}^K(C_j)$  possesses a Lipschitz smooth boundary. We know from equation (15) above that  $\partial\omega_{c_j}^K(C_j) \subset \mathcal{M}$ . There exists a countable (possibly finite) collection of sets  $\{W_i\}_{i \in \Lambda}$ ,  $\Lambda \subseteq \mathbb{N}$ , and Lipschitz maps  $\phi_i : W_i \rightarrow \mathcal{X}$  with Lipschitz continuous inverse  $\phi_i^{-1}$ , such that  $\mathcal{M} = \bigcup_{i \in \Lambda} (\phi_i(W_i) \cap \mathcal{M})$ . We claim

$$(16) \quad \partial\omega_{c_j}^K(C_j) = \bigcup_{i \in \Lambda} \phi_i(\widehat{W}_i)$$

for some countable collection of sets  $\{\widehat{W}_i\}_{i \in \Lambda}$ . Indeed,

$$\begin{aligned} \partial\omega_{c_j}^K(C_j) &= \partial\omega_{c_j}^K(C_j) \cap \mathcal{M} \\ &= \partial\omega_{c_j}^K(C_j) \cap \left( \bigcup_{i \in \Lambda} (\phi_i(W_i) \cap \mathcal{M}) \right) = \bigcup_{i \in \Lambda} (\phi_i(W_i) \cap \partial\omega_{c_j}^K(C_j)). \end{aligned}$$

Next, for each  $i \in \Lambda$ , define the collection of  $(N-1)$ -dimensional sets

$$\widehat{W}_i := \phi_i^{-1} \left( \phi_i(W_i) \cap \partial \omega_{c_j}^K(C_j) \right).$$

This shows (16) and proves that the set  $\omega_{c_j}^K(C_j)$  possess a Lipschitz smooth boundary.  $\square$

*Hypothesis 3.10.* For  $j = 0, 1, 2$ , assume that there is a closed bounded positively invariant set  $G_{j+1}$  in  $\mathcal{X}_{j+1}$ , absorbing relative to all of  $\mathcal{X}_{j+1}$ .

**3.3. The construction (part 2).** Let  $\tau_{j+1} \geq 0$  be the “time of entry,” in  $\mathcal{X}_{j+1}$ , of the bounded set  $\omega_{c_j}^K(C_j)$  into the absorbing set  $G_{j+1}$  of hypothesis 3.10 ( $\tau_{j+1}$  depends on  $\omega_{c_j}^K(C_j)$ ); i.e., for all  $\tau \geq \tau_{j+1}$ ,

$$S(\tau) \omega_{c_j}^K(C_j) \subseteq G_{j+1}.$$

Set  $I_{j+1} = [\tau_{j+1}, 2\tau_{j+1}]$ , and define

$$(17) \quad \widehat{\mathcal{M}}_{j+1} := \bigcup_{\tau \in I_{j+1}} S(\tau) \omega_{c_j}^K(C_j) = \bigcup_{\tau \in I_{j+1}} S(\tau) \left( \bigcap_{s \geq c_j} \overline{\bigcup_{t \geq s} K(t) C_j}^{\mathcal{X}_{j+1}} \right).$$

We claim that, for each  $j = 0, 1, 2$ , this set is a compact inertial manifold in  $\mathcal{X}$ .

*Theorem 3.11.* Suppose hypotheses 3.2, 3.5, 3.6, 3.8, and 3.10 hold. The sets  $\widehat{\mathcal{M}}_{j+1}$  defined by (17) are compact inertial manifolds for  $S$  in  $\mathcal{X}$ , with the following exception:  $\widehat{\mathcal{M}}_{j+1}$  attracts exponentially all bounded subsets of  $\mathcal{X}_{j+1}$  (rather than  $\mathcal{X}$ ) in the topology of  $\mathcal{X}$ .

*Proof.* We need to show that the set  $\widehat{\mathcal{M}}_{j+1}$  is non-empty, compact in  $\mathcal{X}$ , positively invariant under  $S$ , exponentially attracts all bounded subsets of  $\mathcal{X}_{j+1}$  in the topology of  $\mathcal{X}$ , as well as possesses finite fractal dimension, and a Lipschitz smooth boundary. Without repeating,  $j = 0, 1, 2$ .

1. By the compactness of the imbedding  $\mathcal{X}_{j+1} \hookrightarrow \mathcal{X}$ , it follows that  $\widehat{\mathcal{M}}_{j+1}$  is compact in  $\mathcal{X}$ .

2. The fact that the sets  $\widehat{\mathcal{M}}_{j+1}$  are non-empty follows from lemma 3.9.

3. Next we show that  $\widehat{\mathcal{M}}_{j+1}$  is positively invariant under  $S$ . By lemma 3.9, the set  $\omega_{c_j}^K(C_j)$  is positively invariant; hence,

$$\begin{aligned} S(\sigma) \widehat{\mathcal{M}}_{j+1} &= S(\sigma) \bigcup_{\tau \in I_{j+1}} S(\tau) \omega_{c_j}^K(C_j) = \bigcup_{\tau \in I_{j+1}} S(\tau) \left( S(\sigma) \omega_{c_j}^K(C_j) \right) \\ &\subseteq \bigcup_{\tau \in I_{j+1}} S(\tau) \omega_{c_j}^K(C_j) = \widehat{\mathcal{M}}_{j+1}. \end{aligned}$$

This shows  $S(\sigma) \widehat{\mathcal{M}}_{j+1} \subseteq \widehat{\mathcal{M}}_{j+1}$  for all  $\sigma \geq 0$ .

We now show that  $\widehat{\mathcal{M}}_{j+1}$  exponentially attracts all bounded subsets of  $\mathcal{X}_{j+1}$  (in the topology of  $\mathcal{X}$ ).

4. Let  $B_{j+1}$  be a bounded subset of  $\mathcal{X}_{j+1}$ . We need to show that there are  $\kappa, \eta \geq 0$ , depending on  $B_{j+1}$ , such that for all  $t \geq 0$ ,

$$(18) \quad \partial_{\mathcal{X}}(S(t)B_{j+1}, \widehat{\mathcal{M}}_{j+1}) \leq \kappa e^{-\eta t}.$$

The following is borrowed from [24, Prop. 5.8].

The first step is to show that (18) holds when  $B_{j+1} = G_{j+1}$  ( $G_{j+1}$  is the absorbing set given in hypothesis 3.10). In light of the exponential tracking property given in (7), observe that the function  $\mathcal{X} \ni x \mapsto \|Q_N x - m(P_N x)\|_{\mathcal{X}}$  is continuous, so its restriction to the compact set  $G_{j+1}$  is bounded. Thus, there is  $K > 0$ , depending on  $G_{j+1}$ , and there is  $\eta > 0$ , such that for all  $g \in G_{j+1}$ , there is  $\hat{g} \in \mathcal{M}$ , depending on  $g$ , with the property that for all  $t \geq 0$ ,

$$(19) \quad \|S(t)g - S(t)\hat{g}\|_{\mathcal{X}} \leq K e^{-\eta t}.$$

Let  $\widehat{G}_{j+1}$  be the subset of  $\widehat{\mathcal{M}}_{j+1}$  consisting of all the elements  $\hat{g} \in \widehat{\mathcal{M}}_{j+1}$  satisfying (19) for some  $g \in G_{j+1}$ . The set  $\widehat{G}_{j+1}$  is non-empty because when  $g \in \widehat{\mathcal{M}}_{j+1} \subseteq G_{j+1}$ , those  $\hat{g} = g \in \widehat{\mathcal{M}}_{j+1}$  satisfy (19). Since  $\widehat{G}_{j+1} \subseteq \widehat{\mathcal{M}}_{j+1}$  and  $\widehat{\mathcal{M}}_{j+1}$  is positively invariant, then for all  $t \geq 0$ ,  $S(t)\widehat{G}_{j+1} \subseteq \widehat{\mathcal{M}}_{j+1}$ . Hence, (19) implies that for fixed  $\hat{g} \in \widehat{G}_{j+1}$ , and for all  $g \in G_{j+1}$  and  $t \geq 0$ ,

$$(20) \quad \partial_{\mathcal{X}}(S(t)g, \widehat{\mathcal{M}}_{j+1}) \leq \partial_{\mathcal{X}}(S(t)g, S(t)\widehat{G}_{j+1}) \leq \|S(t)g - S(t)\hat{g}\|_{\mathcal{X}} \leq K e^{-\eta t}.$$

Since the right hand side of (20) is independent of  $g$ , it follows that for all  $t \geq 0$ ,

$$(21) \quad \partial_{\mathcal{X}}(S(t)G_{j+1}, \widehat{\mathcal{M}}_{j+1}) \leq K e^{-\eta t}.$$

Now we treat the general case where  $B_{j+1}$  is any bounded subset of  $\mathcal{X}_{j+1}$ . Since  $G_{j+1}$  is absorbing, there is  $t_0 \geq 0$ , depending on  $B_{j+1}$ , such that for all  $t \geq t_0$ ,  $S(t)B_{j+1} \subseteq G_{j+1}$ . Let  $t \geq t_0$ , and write  $t = t_0 + r$ ,  $r \geq 0$ . Since,

$$S(t)B_{j+1} = S(r)S(t_0)B_{j+1} \subseteq S(r)G_{j+1},$$

we have that (with the aid of (21)) when  $t \geq t_0$ ,

$$(22) \quad \begin{aligned} \partial_{\mathcal{X}}(S(t)B_{j+1}, \widehat{\mathcal{M}}_{j+1}) &\leq \partial_{\mathcal{X}}(S(r)G_{j+1}, \widehat{\mathcal{M}}_{j+1}) \leq K e^{-\eta r} \\ &= K e^{-\eta(t-t_0)} = K e^{\eta t_0} e^{-\eta t}. \end{aligned}$$

When  $0 \leq t \leq t_0$ , we estimate

$$(23) \quad \begin{aligned} \partial_{\mathcal{X}}(S(t)B_{j+1}, \widehat{\mathcal{M}}_{j+1}) &\leq \max_{t \in [0, t_0]} \partial_{\mathcal{X}}(S(t)B_{j+1}, \widehat{\mathcal{M}}_{j+1}) =: c \\ &= c e^{\eta t} e^{-\eta t} \leq c e^{\eta t_0} e^{-\eta t}. \end{aligned}$$

So (18) follows from (22) and (23) with  $\kappa := e^{\eta t_0} \max\{c, K\}$ .

It remains to show that  $\widehat{\mathcal{M}}_{j+1}$  possesses finite fractal dimension and a Lipschitz smooth boundary.

5. By applying  $\bigcup_{\tau \in I_{j+1}} S(\tau)$  to (15) of lemma 3.9, and using hypothesis 3.8 and the fact that  $\mathcal{M}$  is positively invariant, it follows that  $\widehat{\mathcal{M}}_{j+1} \subseteq \mathcal{M}$  where we already know  $\mathcal{M}$  has finite (fractal) dimension. It follows that  $\dim_F(\widehat{\mathcal{M}}_{j+1}) \leq \dim_F(\mathcal{M}) = N$  since  $\widehat{\mathcal{M}}_{j+1}$  is compact in  $\mathcal{M}$ .

6. Finally, we show that the set  $\widehat{\mathcal{M}}_{j+1}$  possess Lipschitz smooth boundary. By the very definition

$$\partial \widehat{\mathcal{M}}_{j+1} = \partial \left( \bigcup_{\tau \in I_{j+1}} S(\tau) \omega_{c_j}^K(C_j) \right) = \bigcup_{\tau \in I_{j+1}} S(\tau) \partial \omega_{c_j}^K(C_j),$$

and with the aid of lemma 3.9, we have that

$$\begin{aligned}\partial\widehat{\mathcal{M}}_{j+1} &= \bigcup_{\tau \in I_{j+1}} S(\tau) \partial\omega_{c_j}^K(C_j) = \bigcup_{\tau \in I_{j+1}} S(\tau) \bigcup_{i \in \Lambda} \phi_i(\widehat{W}_i) \\ &= \bigcup_{i \in \Lambda} \left[ \bigcup_{\tau \in I_{j+1}} S(\tau) \phi_i(\widehat{W}_i) \right].\end{aligned}$$

Since the two maps  $S(\tau)$  and  $\phi_i$  are Lipschitz, so is their composition. Thus  $\widehat{\mathcal{M}}_{j+1}$  possess a Lipschitz boundary.

The proof is complete.  $\square$

#### 4. SINGULAR ROBUSTNESS

Let  $Y_0, Y_1, Y_2$ , and  $Y_3$  be Hilbert spaces satisfying

$$Y_3 \subset Y_2 \subset Y_1 \subset Y_0$$

with continuous and compact imbeddings. Additionally assume  $Y_1 \subset Y_0$  densely. Define

$$X_k = Y_k \times Y_{k-1}, \quad k = 1, 2, 3.$$

The spaces  $X_k$  are Hilbert when endowed with the norm whose square is given by

$$\|(u, v)\|_{X_k}^2 = \|u\|_{Y_k}^2 + \|v\|_{Y_{k-1}}^2, \quad k = 1, 2, 3.$$

With an abuse of notation, denote by

$$\Pi_1 : X_k \rightarrow Y_k \text{ and } \Pi_2 : X_k \rightarrow Y_{k-1},$$

the continuous *projections* of  $X_k$  onto its first component  $Y_k$  and, respectively, second component  $Y_{k-1}$ . For  $\varepsilon \in (0, 1]$ , consider in  $X_k$  the equivalent  $\varepsilon$ -weighted norm whose square is given by

$$\|u\|_{Y_k}^2 + \varepsilon \|v\|_{Y_{k-1}}^2,$$

and call  $X_k^\varepsilon$  the corresponding normed space. For  $\varepsilon = 0$ , we set  $X_k^0 = Y_k$  with the norm of  $Y_k$ . We consider a family  $(S_\varepsilon)_{\varepsilon \geq 0}$  of semiflows, respectively on  $X_1^\varepsilon$  when  $\varepsilon > 0$ , and on  $X_1^0 = Y_1$  when  $\varepsilon = 0$ . Observe that  $S_0$  is a semiflow on  $X_1^0 = Y_1$ , not  $Y_0$ . For this reason, we further denote  $Y_0 = X_p^0$  and consider a semiflow  $S_p$  on  $Y_0$ .

Recall the Hausdorff semidistance, in the topology of  $X_k^\varepsilon$ , between two subsets  $U$  and  $V$  of  $X_k^\varepsilon$  is given by

$$\partial_{X_k^\varepsilon}(U, V) := \sup_{x \in U} \inf_{y \in V} \|x - y\|_{X_k^\varepsilon}.$$

Define the *symmetric Hausdorff distance* between  $U$  and  $V$  by

$$\text{dist}_{X_k^\varepsilon}(U, V) := \max\{\partial_{X_k^\varepsilon}(U, V), \partial_{X_k^\varepsilon}(V, U)\}.$$

A family of subsets  $(W^\varepsilon)_{\varepsilon \in [0, 1]}$  of  $X_k^\varepsilon$  is called *robust at*  $\varepsilon_0 \in [0, 1]$  whenever there are constants  $\Lambda > 0$  and  $\phi \in (0, 1]$ , both independent of  $\varepsilon$  and  $\varepsilon_0$ , such that

$$\text{dist}_{X_k^\varepsilon}(W^\varepsilon, W^{\varepsilon_0}) \leq \Lambda |\varepsilon - \varepsilon_0|^\phi.$$

Throughout the remainder of this section we will be interested in *singular robustness*; i.e., robustness at  $\varepsilon_0 = 0$ . One of the problems we need to overcome when we compare a set in  $Y_0$  with one in  $X_1$  is the obvious fact that the topologies are incompatible. To this end, we need to lift sets from  $Y_0$  into the product structure of  $X_1$ , and there we may compare the two structures in the topology of  $X_1^\varepsilon$ .

Before we proceed, we must reassign some notation so that the singular and non-singular cases are compatible with the hypotheses made so far. We introduce the family of initial value problems on  $X_1^\varepsilon$  of the form (1) where now, for  $\varepsilon \in [0, 1]$ ,

$$(24) \quad \begin{cases} \partial_t U_\varepsilon(t) + A_\varepsilon U_\varepsilon(t) = F_\varepsilon(U_\varepsilon(t)), & t > 0 \\ U_\varepsilon(0) = U_\varepsilon^0, & U_\varepsilon^0 \in X_1^\varepsilon. \end{cases}$$

For the singular case ( $\varepsilon = 0$ ): Observe that hypothesis 3.5 is compatible with the new framework given above with

$$\mathcal{X} = Y_0, \quad \mathcal{X}_1 = Y_1, \quad \mathcal{X}_2 = Y_2, \quad \text{and} \quad \mathcal{X}_3 = Y_3.$$

Assume hypotheses 2.1 and 2.2 hold for the IVP (24) with  $\varepsilon = 0$  in place of (1). We then set

$$S_p := S, \quad N_p := N, \quad N_{p*} := N_*,$$

$$\mathfrak{I}_1 := \Sigma_1, \quad \mathfrak{I}_2 := \Sigma_2, \quad m_p := m, \quad \Gamma^p := \Gamma, \quad \text{and} \quad \mathcal{M}_p^0 := \mathcal{M}.$$

Assume hypothesis 3.2 holds and set

$$Z_p := Z \quad \text{and} \quad K_p := K,$$

and assuming hypothesis 3.6 holds, set

$$\mathcal{W}^p := \mathcal{W}, \quad c_i^p := c_i, \quad \text{and} \quad C_i^0 := C_i, \quad i = 0, 1, 2, 3.$$

In accordance with hypothesis 3.8 assume that for all  $t \geq 0$ ,

$$(25) \quad S_p(t)Y_i \subseteq Y_i, \quad i = 0, 1, 2, 3.$$

Assume hypothesis 3.10 holds and set

$$G_p^0 := G \quad \text{and} \quad G_{j+1}^0 := G_{j+1}, \quad j = 0, 1, 2.$$

For the non-singular case ( $\varepsilon > 0$ ): Hypothesis 3.5 is compatible with the above framework when we set

$$\mathcal{X}_1 = X_1^\varepsilon, \quad \mathcal{X}_2 = X_2^\varepsilon, \quad \text{and} \quad \mathcal{X}_3 = X_3^\varepsilon,$$

(when  $\varepsilon > 0$ ,  $\mathcal{X}$  is not needed). Assume hypotheses 2.1 and 2.2 (this time without parts 3 & 4) hold for the IVP (24) with  $\varepsilon \in (0, 1]$  in place of (1); set

$$S_\varepsilon := S, \quad N_\varepsilon := N, \quad N_{\varepsilon*} := N_*, \quad m_\varepsilon := m, \quad \Gamma^\varepsilon := \Gamma, \quad \text{and} \quad \mathcal{M}_1^\varepsilon := \mathcal{M},$$

(we keep  $\Sigma_1$  and  $\Sigma_2$  as they are). Assume hypothesis 3.2 and set

$$Z_\varepsilon := Z \quad \text{and} \quad K_\varepsilon := K,$$

and assuming hypothesis 3.6 holds, set

$$\mathcal{W}^\varepsilon := \mathcal{W}, \quad c_k^\varepsilon := c_k, \quad \text{and} \quad C_k^\varepsilon := C_{k-1} \quad k = 1, 2, 3.$$

In accordance with hypothesis 3.8 assume that for all  $t \geq 0$ ,

$$S_\varepsilon(t)X_k \subseteq X_k, \quad k = 1, 2, 3.$$

Assume hypothesis 3.10 holds and set

$$G_1^\varepsilon := G, \quad G_2^\varepsilon := G_2, \quad \text{and} \quad G_3^\varepsilon := G_3.$$

*Hypothesis 4.1.* Assume there exists a *canonical extension* map  $\mathcal{E} : Y_2 \rightarrow Y_0$  which is locally Lipschitz continuous in  $Y_2$ , and for  $\varepsilon > 0$  the *lift* map  $\mathcal{L} : Y_2 \rightarrow Y_2 \times Y_0$  defined by  $u \mapsto (u, \mathcal{E}u)$  is such that the following holds: there exists  $\gamma \in (0, 1]$  and an interval  $I$ , both independent of  $\varepsilon$ , such that for any bounded set  $B_3$  in  $X_3$ , there exists  $C \geq 0$ , depending on  $B_3$ , but independent of  $\varepsilon$ , such that for all  $t \in I$  and  $x \in B_3$ ,

$$(26) \quad \|S_\varepsilon(t)x - \mathcal{L}S_p(t)\Pi_1^3x\|_{X_1^\varepsilon} \leq C\varepsilon^\gamma.$$

Recall from (25),  $S_p(t)\Pi_1^3x \in Y_3 \subset Y_2$ ; i.e.,  $S_p(t)\Pi_1^3x$  is in the domain of the lift operator.

Now we construct the compact inertial manifolds for the singular and non-singular cases; recall the singular and non-singular cases of hypotheses 2.1, 2.2, 3.2, 3.5, 3.6, 3.8, and 3.10, were described above.

First we construct  $\widehat{\mathcal{M}}_3^0$  (here  $\varepsilon = 0$ ): Let  $\mathcal{W}^p$  be a bounded subset of  $\Gamma^p \subseteq \mathfrak{I}_1$  and define the bounded set of  $\mathcal{M}_p^0$ ,

$$\mathcal{M}_{\mathcal{W}^p}^0 := \text{graph}|_{\mathcal{W}^p}(m_p) = \{\xi + m_p(\xi) : \xi \in \mathcal{W}^p\} \text{ (cf. (9)).}$$

Set

$$C_0^0 := \mathcal{M}_{\mathcal{W}^p}^0, \quad C_1^0 := \bigcup_{t \geq c_0^p} K_p(t)C_0^0, \quad C_2^0 := \bigcup_{t \geq c_1^p} K_p(t)C_1^0,$$

and

$$\omega_{c_2^p}^{K_p}(C_2^0) := \bigcap_{s \geq c_2^p} \overline{\bigcup_{t \geq s} K_p(t)C_2^0}^{Y_3} \text{ (cf. (14)).}$$

The set  $\omega_{c_2^p}^{K_p}(C_2^0)$  is non-empty, compact in  $Y_0$ , positively invariant under  $S_p$ , possesses a Lipschitz smooth boundary, and satisfies

$$\omega_{c_2^p}^{K_p}(C_2^0) \subseteq \mathcal{M}_p^0 \cap Y_3 \text{ (cf. (15) of lemma 3.9).}$$

Let  $\tau_3^p \geq 0$  be such that for all  $\tau \geq \tau_3^p$ ,

$$S(\tau)\omega_{c_2^p}^{K_p}(C_2^0) \subseteq G_3^0,$$

and set  $I_3^p := [\tau_3^p, 2\tau_3^p]$ . Finally, the set

$$(27) \quad \widehat{\mathcal{M}}_3^0 := \bigcup_{\tau \in I_3^p} S_p(\tau)\omega_{c_2^p}^{K_p}(C_2^0) \text{ (cf. (17))}$$

is a compact inertial manifold for  $S_p$  in  $Y_0$ .

Define the compact inertial manifold  $\widehat{\mathcal{M}}_3^\varepsilon$  as follows: Given a bounded subset  $\mathcal{W}^\varepsilon$  of  $\Gamma^\varepsilon$ , define the bounded set of  $\mathcal{M}_1^\varepsilon$ ,

$$\mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon := \text{graph}|_{\mathcal{W}^\varepsilon}(m_\varepsilon) = \{\xi + m_\varepsilon(\xi) : \xi \in \mathcal{W}^\varepsilon\} \text{ (cf. (9)).}$$

Set

$$C_1^\varepsilon := \mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon, \quad C_2^\varepsilon := \bigcup_{t \geq c_1^\varepsilon} K_\varepsilon(t)C_1^\varepsilon,$$

and define

$$\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) := \bigcap_{s \geq c_2^\varepsilon} \overline{\bigcup_{t \geq s} K_\varepsilon(t)C_2^\varepsilon}^{X_3} \text{ (cf. (14)).}$$

The set  $\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  is non-empty, compact in  $X_1$ , positively invariant under  $S_\varepsilon$ , possesses a Lipschitz smooth boundary, and satisfies

$$\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \subseteq \mathcal{M}_1^\varepsilon \cap X_3 \text{ (cf. (15) of lemma 3.9).}$$

Let  $\tau_3^\varepsilon \geq 0$  be such that for all  $\tau \geq \tau_3^\varepsilon$ ,

$$S(\tau)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \subseteq G_3^\varepsilon,$$

and set  $I_3^\varepsilon := [\tau_3^\varepsilon, 2\tau_3^\varepsilon]$ . Then the set

$$(28) \quad \widehat{\mathcal{M}}_3^\varepsilon := \bigcup_{\tau \in I_3^\varepsilon} S_\varepsilon(\tau)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \text{ (cf. (17))}$$

is a compact inertial manifold for  $S_\varepsilon$  in  $X_1$ .

*Remark 4.2.* Recall  $Y_0 = \mathfrak{J}_1 \oplus \mathfrak{J}_2$  where  $\mathfrak{J}_1$  is a closed finite dimensional subspace of  $Y_0$  and  $\mathfrak{J}_2 = \mathfrak{J}_1^\perp$ . Since  $Y_1 \subset Y_0$  densely, then  $Y_1 = \mathfrak{J}_1 \oplus \widetilde{\mathfrak{J}}_2$  where

$$\widetilde{\mathfrak{J}}_2 := \{y \in Y_1 : \langle y, z \rangle_{Y_1} = 0, \forall z \in \mathfrak{J}_1\}$$

(i.e., the product is in  $Y_1$  rather than  $Y_0$ ). With this we claim, for all  $\varepsilon \in (0, 1]$ ,

$$(29) \quad X_1^\varepsilon = Y_1 \times Y_0 = (\mathfrak{J}_1 \oplus \widetilde{\mathfrak{J}}_2) \times (\mathfrak{J}_1 \oplus \mathfrak{J}_2) = (\mathfrak{J}_1 \times \mathfrak{J}_1) \oplus (\widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2).$$

Indeed, any  $x \in X_1$  can be represented as a pair  $x = (u, v)$  for some  $u \in Y_1$  and  $v \in Y_0$ . In turn,  $u = p + q$  and  $v = \chi + \psi$  for some  $p, \chi \in \mathfrak{J}_1$ ,  $q \in \widetilde{\mathfrak{J}}_2$ , and  $\psi \in \mathfrak{J}_2$ . Hence,

$$(p + q, \chi + \psi) = (p, \chi) + (q, \psi) \in (\mathfrak{J}_1 \times \mathfrak{J}_1) \oplus (\widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2).$$

On the other hand, let  $(a_1, a_2) \in \mathfrak{J}_1 \times \mathfrak{J}_1$  and  $(b_1, b_2) \in \widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2$ . Then

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) = (c, d)$$

where  $c = a_1 + b_1 \in Y_1$  and  $d = a_2 + b_2 \in Y_0$ . Thus,  $(c, d) \in X_1$  and (29) holds.

*Lemma 4.3.* Suppose  $\Sigma_1 = \mathfrak{J}_1 \times \mathfrak{J}_1$  as in (29). Then  $\Sigma_1^\perp = \widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2$ .

*Proof.* First, let

$$x \in \Sigma_1^\perp = \{y \in X_1 : \langle y, z \rangle_{X_1} = 0, \forall z \in \Sigma_1\}.$$

Since  $x \in X_1$ , by (29)  $x = (\chi_1 + \psi_1, \chi_2 + \psi_2)$  for some  $\chi_1, \chi_2 \in \mathfrak{J}_1$ ,  $\psi_1 \in \widetilde{\mathfrak{J}}_2$ , and  $\psi_2 \in \mathfrak{J}_2$ . Let  $\xi_1, \xi_2 \in \mathfrak{J}_1$  and  $z = (\xi_1, \xi_2)$ . Then  $\langle x, z \rangle_{X_1} = 0$ , which implies

$$\langle \chi_1 + \psi_1, \xi_1 \rangle_{Y_1} + \langle \chi_2 + \psi_2, \xi_2 \rangle_{Y_0} = 0.$$

Since  $\xi_1, \xi_2 \in \mathfrak{J}_1$ , while  $\psi_1 \in \widetilde{\mathfrak{J}}_2 = \mathfrak{J}_1^\perp$  in  $Y_1$  and  $\psi_2 \in \mathfrak{J}_2 = \mathfrak{J}_1^\perp$  in  $Y_0$ , it follows that both

$$\langle \psi_1, \xi_1 \rangle_{Y_1} = 0 \text{ and } \langle \psi_2, \xi_2 \rangle_{Y_0} = 0.$$

This leaves us with

$$\langle \chi_1, \xi_1 \rangle_{Y_1} + \langle \chi_2, \xi_2 \rangle_{Y_0} = 0; \text{ i.e. } \langle (\chi_1, \chi_2), (\xi_1, \xi_2) \rangle_{X_1} = 0 \quad \forall z = (\xi_1, \xi_2) \in \Sigma_1;$$

that is,  $(\chi_1, \chi_2) \in \Sigma_1^\perp$ . But since  $(\chi_1, \chi_2) \in \mathfrak{J}_1 \times \mathfrak{J}_1 = \Sigma_1$ , it follows that  $\chi_1 = \chi_2 = 0$ . Thus,  $x = (\psi_1, \psi_2) \in \widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2$ ; that is,  $\Sigma_1^\perp \subseteq \widetilde{\mathfrak{J}}_2 \times \mathfrak{J}_2$ . The other direction is analogous.  $\square$

*Lemma 4.4.* Suppose  $\Sigma_1 = \mathfrak{J}_1 \times \mathfrak{J}_1$  as in (29) and set  $\Sigma_2 = \Sigma_1^\perp$ . For  $L > 0$ , define

$$\tilde{\mathcal{G}}_L^0 := \left\{ \Psi \in C_b(\mathfrak{J}_1; \tilde{\mathfrak{J}}_2) : \|\Psi(\chi_1) - \Psi(\psi_1)\|_{Y_1} \leq L\|\chi_1 - \psi_1\|_{Y_1}, \forall \chi_1, \psi_1 \in \mathfrak{J}_1 \right\},$$

$$\mathcal{G}_L^0 := \left\{ \Psi \in C_b(\mathfrak{J}_1; \mathfrak{J}_2) : \|\Psi(\chi_2) - \Psi(\psi_2)\|_{Y_0} \leq L\|\chi_2 - \psi_2\|_{Y_0}, \forall \chi_2, \psi_2 \in \mathfrak{J}_1 \right\},$$

and

$$\mathcal{G}_L^\varepsilon := \left\{ \Psi \in C_b(\Sigma_1; \Sigma_2) : \|\Psi(\chi) - \Psi(\psi)\|_{X_1} \leq L\|\chi - \psi\|_{X_1}, \forall \chi, \psi \in \Sigma_1 \right\}.$$

$$\text{Then } \tilde{\mathcal{G}}_L^0 \times \mathcal{G}_L^0 = \mathcal{G}_L^\varepsilon.$$

*Proof.* Let  $L > 0$  be given and  $g \in \tilde{\mathcal{G}}_L^0$ ,  $h \in \mathcal{G}_L^0$ . For all  $\chi = (\chi_1, \chi_2) \in \Sigma_1$  define

$$f(\chi) := (g(\chi_1), h(\chi_2)).$$

After identifying

$$C_b(\mathfrak{J}_1; \tilde{\mathfrak{J}}_2) \times C_b(\mathfrak{J}_1; \mathfrak{J}_2) \equiv C_b(\mathfrak{J}_1 \times \tilde{\mathfrak{J}}_2; \mathfrak{J}_1 \times \mathfrak{J}_2),$$

then  $f \in C_b(\Sigma_1; \Sigma_2)$ .

Now let  $\chi = (\chi_1, \chi_2)$  and  $\psi = (\psi_1, \psi_2) \in \Sigma_1$ . Then

$$\begin{aligned} \|f(\chi) - f(\psi)\|_{X_1}^2 &= \|g(\chi_1) - g(\psi_1)\|_{Y_1}^2 + \|h(\chi_2) - h(\psi_2)\|_{Y_0}^2 \\ &\leq L^2\|\chi_1 - \psi_1\|_{Y_1}^2 + L^2\|\chi_2 - \psi_2\|_{Y_0}^2 \leq L^2\|\chi - \psi\|_{X_1}^2. \end{aligned}$$

This shows  $\tilde{\mathcal{G}}_L^0 \times \mathcal{G}_L^0 \subseteq \mathcal{G}_L^\varepsilon$ . The other direction is similar: let  $f \in \mathcal{G}_L^\varepsilon$  and define  $g := \Pi_1^1 f \in C_b(\mathfrak{J}_1; \tilde{\mathfrak{J}}_2)$  and  $h := \Pi_2^1 f \in C_b(\mathfrak{J}_1; \mathfrak{J}_2)$ .  $\square$

Before we continue, special care needs to be made concerning the “size” of the compact inertial manifolds under consideration. Assume that  $\widehat{\mathcal{M}}_3^0$  was constructed as above. We now construct  $\widehat{\mathcal{M}}_3^\varepsilon$  as above but in accordance with the following modifications; these are the *compatibility criteria*:

- (1) Define  $N^* := \max\{N_{p*}, N_{\varepsilon*}\}$ . Then define  $\mathfrak{J}_1 := \text{span}\{\omega_n\}_{n=1}^{N^*}$  and  $\Sigma_1 := \mathfrak{J}_1 \times \mathfrak{J}_1$ .
- (2) Given  $\mathcal{W}^p \subset \mathfrak{J}_1$ , define  $\mathcal{W}^\varepsilon := \mathcal{W}^p \times \mathcal{W}^p \subset \Sigma_1$ . It follows that  $\dim(\mathfrak{J}_1) = \dim(\Sigma_1) = N^*$ .
- (3) Define  $\tau_3^* := \max\{\tau_3^p, \tau_3^\varepsilon\}$ . Replace  $\tau_3^p$  and  $\tau_3^\varepsilon$  with  $\tau_3^*$ . Also replace  $I_3^p$  and  $I_3^\varepsilon$  with  $I_3^* := [\tau_3^*, 2\tau_3^*]$ .
- (4) Define  $c_1^* := \max\{c_1^p, c_1^\varepsilon\}$ , and replace  $c_1^p$  and  $c_1^\varepsilon$  with  $c_1^*$ . Also define  $c_2^* := \max\{c_2^p, c_2^\varepsilon\}$ , and replace  $c_2^p$  and  $c_2^\varepsilon$  with  $c_2^*$ .

*Hypothesis 4.5.* When  $\varepsilon = 0$ , assume that the decomposition in hypothesis 3.2 satisfies  $S_p = K_p$ . This is a regularity assumption on the semiflow  $S_p$ .

*Hypothesis 4.6.* The final hypothesis we make is the following: Let  $m_p \in \mathcal{G}_L^0$  and  $m_\varepsilon = (f, g) \in \mathcal{G}_L^\varepsilon = \tilde{\mathcal{G}}_L^0 \times \mathcal{G}_L^0$ . Assume  $f \in \tilde{\mathcal{G}}_L^0$  is such that the following holds: there exist  $C, C' > 0$  and  $\lambda, \lambda' \in (0, 1]$ , depending on  $f$  and  $m_p$  but independent of  $\varepsilon$ , such that for all  $\sigma_0 \geq \tau_3^p + c_2^p + c_1^p$  and  $\sigma_1 \geq \tau_3^p + c_2^p + c_1^p + c_0^p$ ,

$$(30) \quad \sup_{\chi \in \mathcal{W}^p} \|S_p(\sigma_0)f(\chi) - S_p(\sigma_0)m_p(\chi)\|_{Y_1} \leq C\varepsilon^\lambda,$$

and

$$(31) \quad \sup_{\zeta \in \mathcal{W}^p} \|S_p(\sigma_1)m_p(\zeta) - S_p(\sigma_1)f(\zeta)\|_{Y_1} \leq C'\varepsilon^{\lambda'}.$$

Finally we describe the singular robustness problem: Our goal is to study the singular robustness (i.e., robustness at  $\varepsilon_0 = 0$ ) of the sets  $\widehat{\mathcal{M}}_3^\varepsilon$ ,  $\varepsilon \in (0, 1]$ , with the set  $\widehat{\mathcal{M}}_3^0$ . Our main question now is:

$$\text{"In what sense does } \widehat{\mathcal{M}}_3^\varepsilon \rightarrow \widehat{\mathcal{M}}_3^0 \text{ as } \varepsilon \rightarrow 0?"$$

Recall that for each  $\varepsilon \in (0, 1]$ , the set  $\widehat{\mathcal{M}}_3^\varepsilon$  is in the product space  $X_3 = Y_3 \times Y_2$ ; however, the limit set  $\widehat{\mathcal{M}}_3^0$  is in  $Y_3$ . We know that upon lifting  $\widehat{\mathcal{M}}_3^0$  from  $Y_2$  into  $Y_2 \times Y_0$  via the map  $\mathcal{L}$ , we may then investigate distances between the two subsets in  $X_1$ , and whether  $\widehat{\mathcal{M}}_3^\varepsilon \rightarrow \mathcal{L}\widehat{\mathcal{M}}_3^0$  as  $\varepsilon \rightarrow 0$ . Observe, the indicated convergence is in the topology of  $X_1^\varepsilon$ . This is important: the sets  $\widehat{\mathcal{M}}_3^\varepsilon$  and  $\mathcal{L}\widehat{\mathcal{M}}_3^0$  are in  $X_1$ . The space  $X_1$  is endowed with the equivalent topology of  $X_1^\varepsilon$ , and it is here that we study the convergence of  $\widehat{\mathcal{M}}_3^\varepsilon \rightarrow \mathcal{L}\widehat{\mathcal{M}}_3^0$ . We will see why  $\widehat{\mathcal{M}}_3^0$  needs to be in  $Y_3$  later in the application.

**4.1. The main theorem.** Define the family of sets  $(\mathbb{M}_3^\varepsilon)_{\varepsilon \in [0, 1]}$  in  $X_1^\varepsilon$  by

$$(32) \quad \mathbb{M}_3^\varepsilon = \begin{cases} \widehat{\mathcal{M}}_3^\varepsilon & \text{for } 0 < \varepsilon \leq 1 \\ \mathcal{L}\widehat{\mathcal{M}}_3^0 & \text{when } \varepsilon = 0. \end{cases}$$

The main result of this article is to show that there exist  $\Lambda > 0$  and  $\phi \in (0, 1]$ , both independent of  $\varepsilon$ , so that

$$\text{dist}_{X_1^\varepsilon}(\mathbb{M}_3^\varepsilon, \mathbb{M}_3^0) \leq \Lambda \varepsilon^\phi;$$

i.e., the families of sets in (32) are robust at  $\varepsilon_0 = 0$ . Note that we are not claiming that  $\mathcal{L}\widehat{\mathcal{M}}_3^0$  is a compact inertial manifold for  $S_\varepsilon$  in  $X_1$ . When we speak of a family of compact inertial manifolds, we mean the family of sets  $\widehat{\mathcal{M}}_3^\varepsilon$  in  $X_1$  and  $\widehat{\mathcal{M}}_3^0$  in  $Y_0$ .

*Theorem 4.7.* Let  $\widehat{\mathcal{M}}_3^0$  be the compact inertial manifold constructed above, and for each  $\varepsilon \in (0, 1]$ , let  $\widehat{\mathcal{M}}_3^\varepsilon$  be the compact inertial manifold constructed as above while under the compatibility criteria. If hypotheses 4.1, 4.5, and 4.6 hold, then the family  $(\mathbb{M}_3^\varepsilon)_{\varepsilon \in [0, 1]}$  defined in (32) is robust at  $\varepsilon_0 = 0$  in the topology of  $X_1^\varepsilon$ .

*Proof.* We need to show that there exist  $\Lambda > 0$  and  $\phi \in (0, 1]$ , both independent of  $\varepsilon$ , such that

$$\text{dist}_{X_1^\varepsilon}(\mathbb{M}_3^\varepsilon, \mathbb{M}_3^0) \leq \Lambda \varepsilon^\phi.$$

First we consider the case,

$$(33) \quad \partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) = \sup_{a \in \widehat{\mathcal{M}}_3^\varepsilon} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|a - b\|_{X_1^\varepsilon}.$$

Recall,

$$\widehat{\mathcal{M}}_3^\varepsilon = \bigcup_{\tau \in I_3^\varepsilon} S_\varepsilon(\tau) \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \text{ and } \widehat{\mathcal{M}}_3^0 = \bigcup_{\tau \in I_3^0} S_p(\tau) \omega_{c_2^0}^{K_p}(C_2^0).$$

Fix  $t \in I_3^\varepsilon$  and  $\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  so that  $a = S_\varepsilon(t)\alpha \in \widehat{\mathcal{M}}_3^\varepsilon$ . Then

$$\begin{aligned} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|a - b\|_{X_1^\varepsilon} &= \inf_{\substack{\theta \in I_3^0 \\ \beta \in \omega_{c_2^0}^{K_p}(C_2^0)}} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(\theta)\beta\|_{X_1^\varepsilon} \\ &\leq \inf_{\beta \in \omega_{c_2^0}^{K_p}(C_2^0)} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(\theta)\beta\|_{X_1^\varepsilon} \end{aligned}$$

for all  $\theta \in I_3^p$ . By the compatibility criterion #3, we know  $I_3^p = I_3^\varepsilon$ , so we can choose  $\theta = t$ ; hence,

$$\inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|a - b\|_{X_1^\varepsilon} \leq \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon},$$

and, recall  $S_\varepsilon(t)\alpha = a$ , so

$$\begin{aligned} \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|S_\varepsilon(t)\alpha - b\|_{X_1^\varepsilon} &\leq \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon} \\ &= \partial_{X_1^\varepsilon}(S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon), \mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0)) \\ &\leq \max_{t \in I_3^\varepsilon} \partial_{X_1^\varepsilon}(S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon), \mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0)). \end{aligned}$$

Thus,

$$\sup_{t \in I_3^\varepsilon} \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|S_\varepsilon(t)\alpha - b\|_{X_1^\varepsilon} \leq \max_{t \in I_3^\varepsilon} \partial_{X_1^\varepsilon}(S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon), \mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0)),$$

and

$$\begin{aligned} \sup_{a \in \widehat{\mathcal{M}}_3^\varepsilon} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|a - b\|_{X_1^\varepsilon} &\leq \sup_{t \in I_3^\varepsilon} \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{b \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \|S_\varepsilon(t)\alpha - b\|_{X_1^\varepsilon} \\ (34) \quad &\leq \max_{t \in I_3^\varepsilon} \partial_{X_1^\varepsilon}(S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon), \mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0)) \\ &\leq \max_{t \in I_3^\varepsilon} \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon}. \end{aligned}$$

We now estimate

$$\begin{aligned} (35) \quad \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon} &\leq \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\Pi_1^3\alpha\|_{X_1^\varepsilon} \\ &\quad + \|\mathcal{L}S_p(t)\Pi_1^3\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon}. \end{aligned}$$

By hypothesis 4.1 we have,

$$\|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\Pi_1^3\alpha\|_{X_1^\varepsilon} \leq C_1 \varepsilon^\gamma.$$

On the other hand, by expanding the square of the last norm in (35) we obtain

$$\begin{aligned} \|\mathcal{L}S_p(t)\Pi_1^3\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon}^2 &= \|(S_p(t)\Pi_1^3\alpha, \mathcal{E}S_p(t)\Pi_1^3\alpha) - (S_p(t)\beta, \mathcal{E}S_p(t)\beta)\|_{X_1^\varepsilon}^2 \\ &= \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1}^2 + \varepsilon \|\mathcal{E}S_p(t)\Pi_1^3\alpha - \mathcal{E}S_p(t)\beta\|_{Y_0}^2. \end{aligned}$$

By the local Lipschitz continuity of  $\mathcal{E}$  (hypothesis 4.1), there is  $L > 0$ , depending on  $(\Pi_1^3\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \cup \omega_{c_2^p}^{K_p}(C_2^0)) \subset Y_0$  but independent of  $\varepsilon$ , such that

$$\|\mathcal{E}S_p(t)\Pi_1^3\alpha - \mathcal{E}S_p(t)\beta\|_{Y_0}^2 \leq L^2 \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_2}^2.$$

The sets  $\Pi_1^3\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  and  $\omega_{c_2^p}^{K_p}(C_2^0)$  are compact in  $Y_2$ , so under the continuous image of  $S_p(t)$  we have that

$$\|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_2} \leq \max_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\Pi_1^3\alpha\|_{Y_2} + \max_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\beta\|_{Y_2} \leq C_2$$

for some  $C_2 > 0$  independent of  $\varepsilon$ . Hence,

$$\|\mathcal{L}S_p(t)\Pi_1^3\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon}^2 \leq \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1}^2 + L^2 C_2^2 \varepsilon,$$

and (35) becomes

$$\begin{aligned} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon} &\leq C_1\varepsilon^\gamma + (\|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1}^2 + L^2C_2^2\varepsilon)^{1/2} \\ &\leq C_1\varepsilon^\gamma + \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1} + LC_2\sqrt{\varepsilon} \\ &\leq C_3\varepsilon^{\gamma_0} + \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1} \end{aligned}$$

where  $C_3 := 2 \max\{C_1, LC_2\}$  and  $\gamma_0 := \min\{\frac{1}{2}, \gamma\}$ , and in turn,

$$\begin{aligned} (36) \quad &\sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_\varepsilon(t)\alpha - \mathcal{L}S_p(t)\beta\|_{X_1^\varepsilon} \\ &\leq C_3\varepsilon^{\gamma_0} + \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1}. \end{aligned}$$

Now, using the compatibility criterion #4,

$$\begin{aligned} \Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) &= \Pi_1^1|_{Y_3} \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) = \Pi_1^1|_{Y_3} \bigcap_{s \geq c_2^\varepsilon} \bigcup_{t \geq s} K_\varepsilon(t) \left( \bigcup_{r \geq c_1^\varepsilon} K_\varepsilon(r) \mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon \right) \\ &= \bigcap_{s \geq c_2^p} \bigcup_{t \geq s} K_p(t) \left( \bigcup_{r \geq c_1^p} K_p(r) \Pi_1^1 \mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon \right), \end{aligned}$$

and any  $\Pi_1^3\alpha \in \Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  can be written as  $\Pi_1^3\alpha = K_p(t_0)K_p(r_0)\Pi_1^1(\xi + m_\varepsilon(\xi))$  for some  $t_0 \geq c_2^p$ ,  $r_0 \geq c_1^p$ , and  $\xi \in \mathcal{W}^\varepsilon$ . By the compatibility criterion #2,  $\mathcal{W}^\varepsilon = \mathcal{W}^p \times \mathcal{W}^p$ , so we can write  $\xi = (\chi, \psi)$  for some  $\chi, \psi \in \mathcal{W}^p$ . Also, by lemma 4.4,

$$\xi + m_\varepsilon(\xi) = (\chi, \psi) + m_\varepsilon(\chi, \psi) = (\chi, \psi) + (f(\chi), g(\psi)) = (\chi + f(\chi), \psi + g(\psi))$$

for some  $f \in \tilde{\mathcal{G}}_L^0$  and  $g \in \mathcal{G}_L^0$ . Consequently,

$$\begin{aligned} S_p(t)\Pi_1^3\alpha &= S_p(t)K_p(t_0)K_p(r_0)\Pi_1^1(\chi + f(\chi), \psi + g(\psi)) \\ &= S_p(t)K_p(t_0)K_p(r_0)(\chi + f(\chi)). \end{aligned}$$

In a similar fashion, we can write

$$S_p(t)\beta = S_p(t)K_p(t_1)K_p(r_1)K_p(s_1)(\zeta + m_p(\zeta))$$

for some  $t_1 \geq c_2^p$ ,  $r_1 \geq c_1^p$ ,  $s_1 \geq c_0^p$ , and  $\zeta \in \mathcal{W}^p$ . Using hypothesis 4.5, we have,

$$\begin{aligned} &\|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1} \\ &= \|S_p(t)K_p(t_0)K_p(r_0)(\chi + f(\chi)) - S_p(t)K_p(t_1)K_p(r_1)K_p(s_1)(\zeta + m_p(\zeta))\|_{Y_1} \\ &= \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_1)S_p(s_1)(\zeta + m_p(\zeta))\|_{Y_1} \end{aligned}$$

where  $\sigma_0 := t + t_0 + r_0 \geq \tau_3^p + c_2^p + c_1^p$  and  $\sigma_1 := t + t_1 + r_1 \geq \tau_3^p + c_2^p + c_1^p$ . Since  $\zeta + m_p(\zeta) \in \mathcal{M}_{\mathcal{W}^p}^0 \subset \mathcal{M}_p^0$  and  $\mathcal{M}_p^0$  is positively invariant, then we may express  $S_p(s_1)(\zeta + m_p(\zeta))$  as  $\zeta_1 + m_p(\zeta_1)$  for some  $\zeta_1 \in \Gamma^p$ . Thus,

$$\|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1} = \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_1)(\zeta_1 + m_p(\zeta_1))\|_{Y_1}.$$

Fix  $\sigma_0 \geq \tau_3^p + c_2^p + c_1^p$  and  $\chi \in \mathcal{W}^p$  so that  $\Pi_1^3 \alpha = S_p(\sigma_0)(\chi + f(\chi)) \in \Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$ . Then

$$\begin{aligned} & \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \\ & \leq \inf_{\substack{\sigma_1 \geq \tau_3^p + c_2^p + c_1^p \\ \zeta_1 \in \Gamma^p}} \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_1)(\zeta_1 + m_p(\zeta_1))\|_{Y_1} \\ & \leq \inf_{\zeta_1 \in \Gamma^p} \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_1)(\zeta_1 + m_p(\zeta_1))\|_{Y_1} \end{aligned}$$

for all  $\sigma_1 \geq \tau_3^p + c_2^p + c_1^p$ ; in particular, we can choose  $\sigma_1 = \sigma_0$ ; hence,

$$\inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \leq \inf_{\zeta_1 \in \Gamma^p} \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_0)(\zeta_1 + m_p(\zeta_1))\|_{Y_1}$$

and

$$\begin{aligned} & \sup_{\chi \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \\ & \leq \sup_{\chi \in \mathcal{W}^p} \inf_{\zeta_1 \in \Gamma^p} \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_0)(\zeta_1 + m_p(\zeta_1))\|_{Y_1} \\ & \leq \sup_{\chi \in \mathcal{W}^p} \|S_p(\sigma_0)(\chi + f(\chi)) - S_p(\sigma_0)(\zeta_1 + m_p(\zeta_1))\|_{Y_1} \end{aligned}$$

for all  $\zeta_1 \in \Gamma^p$ ; in particular, we can choose  $\zeta_1 = \chi$ ; so,

$$\sup_{\chi \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \leq \sup_{\chi \in \mathcal{W}^p} \|S_p(\sigma_0)f(\chi) - S_p(\sigma_0)m_p(\chi)\|_{Y_1}.$$

Applying (30) of hypothesis 4.6 yields,

$$\begin{aligned} (37) \quad & \sup_{\chi \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \\ & \leq \sup_{\chi \in \mathcal{W}^p} \|S_p(\sigma_0)f(\chi) - S_p(\sigma_0)m_p(\chi)\|_{Y_1} \leq C_4 \varepsilon^\lambda. \end{aligned}$$

Collecting (33), (34), (36), and (37), yields,

$$\partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq C_3 \varepsilon^{\gamma_0} + C_4 \varepsilon^\lambda,$$

and therefore,

$$(38) \quad \partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq \Lambda_1 \varepsilon^{\phi_1}$$

where  $\Lambda_1 := 2 \max\{C_3, C_4\}$  and  $\phi_1 := \min\{\gamma_0, \lambda\}$ .

The other direction is similar. Indeed,

$$(39) \quad \partial_{X_1^\varepsilon}(\mathcal{L}\widehat{\mathcal{M}}_3^0, \widehat{\mathcal{M}}_3^\varepsilon) = \sup_{a \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|a - b\|_{X_1^\varepsilon}.$$

Once again recall,

$$\widehat{\mathcal{M}}_3^0 = \bigcup_{\tau \in I_3^p} S_p(\tau) \omega_{c_2^p}^{K_p}(C_2^0) \text{ and } \widehat{\mathcal{M}}_3^\varepsilon = \bigcup_{\tau \in I_3^\varepsilon} S_\varepsilon(\tau) \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon).$$

Fix  $t \in I_3^p$  and  $\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)$  so that  $a = \mathcal{L}S_p(t)\alpha \in \mathcal{L}\widehat{\mathcal{M}}_3^0$ . Then

$$\begin{aligned} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|a - b\|_{X_1^\varepsilon} &= \inf_{\substack{\theta \in I_3^\varepsilon \\ \beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)}} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(\theta)\beta\|_{X_1^\varepsilon} \\ &\leq \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(\theta)\beta\|_{X_1^\varepsilon} \end{aligned}$$

for all  $\theta \in I_3^\varepsilon$ . Again, by the compatibility criterion #3, we know  $I_3^\varepsilon = I_3^p$ , so we can choose  $\theta = t$ ; hence,

$$\inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|\mathcal{L}S_p(t)\alpha - b\|_{X_1^\varepsilon} \leq \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon},$$

and

$$\begin{aligned} \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|\mathcal{L}S_p(t)\alpha - b\|_{X_1^\varepsilon} &\leq \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon} \\ &= \partial_{X_1^\varepsilon}(\mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0), S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)) \\ &\leq \max_{t \in I_3^p} \partial_{X_1^\varepsilon}(\mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0), S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)). \end{aligned}$$

Thus,

$$\sup_{t \in I_3^p} \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|\mathcal{L}S_p(t)\alpha - b\|_{X_1^\varepsilon} \leq \max_{t \in I_3^p} \partial_{X_1^\varepsilon}(\mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0), S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)),$$

and

$$\begin{aligned} (40) \quad \sup_{a \in \mathcal{L}\widehat{\mathcal{M}}_3^0} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|a - b\|_{X_1^\varepsilon} &\leq \sup_{t \in I_3^p} \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{b \in \widehat{\mathcal{M}}_3^\varepsilon} \|\mathcal{L}S_p(t)\alpha - b\|_{X_1^\varepsilon} \\ &\leq \max_{t \in I_3^p} \partial_{X_1^\varepsilon}(\mathcal{L}S_p(t)\omega_{c_2^p}^{K_p}(C_2^0), S_\varepsilon(t)\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)) \\ &\leq \max_{t \in I_3^p} \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon}. \end{aligned}$$

We now estimate

$$(41) \quad \begin{aligned} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon} &\leq \|S_\varepsilon(t)\beta - \mathcal{L}S_p(t)\Pi_1^3\beta\|_{X_1^\varepsilon} \\ &\quad + \|\mathcal{L}S_p(t)\Pi_1^3\beta - \mathcal{L}S_p(t)\alpha\|_{X_1^\varepsilon}. \end{aligned}$$

By hypothesis 4.1 we have,

$$\|S_\varepsilon(t)\beta - \mathcal{L}S_p(t)\Pi_1^3\beta\|_{X_1^\varepsilon} \leq C_1\varepsilon^\gamma.$$

On the other hand,

$$\begin{aligned} \|\mathcal{L}S_p(t)\Pi_1^3\beta - \mathcal{L}S_p(t)\alpha\|_{X_1^\varepsilon}^2 &= \|(S_p(t)\Pi_1^3\beta, \mathcal{E}S_p(t)\Pi_1^3\beta) - (S_p(t)\alpha, \mathcal{E}S_p(t)\alpha)\|_{X_1^\varepsilon}^2 \\ &= \|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_1}^2 + \varepsilon\|\mathcal{E}S_p(t)\Pi_1^3\beta - \mathcal{E}S_p(t)\alpha\|_{Y_0}^2. \end{aligned}$$

By the local Lipschitz continuity of  $\mathcal{E}$  (hypothesis 4.1), there is  $L > 0$ , depending on  $(\Pi_1^3\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) \cup \omega_{c_2^p}^{K_p}(C_2^0)) \subset Y_0$  but independent of  $\varepsilon$ , such that

$$\|\mathcal{E}S_p(t)\Pi_1^3\beta - \mathcal{E}S_p(t)\alpha\|_{Y_0}^2 \leq L^2\|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_2}^2.$$

The sets  $\Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  and  $\omega_{c_2^\varepsilon}^{K_p}(C_2^0)$  are compact in  $Y_2$ , so under the continuous image of  $S_p(t)$  we have that

$$\|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_2} \leq \max_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\Pi_1^3\beta\|_{Y_2} + \max_{\alpha \in \omega_{c_2^\varepsilon}^{K_p}(C_2^0)} \|S_p(t)\alpha\|_{Y_2} \leq C'_2$$

for some  $C'_2 > 0$  independent of  $\varepsilon$ . Hence,

$$\|\mathcal{L}S_p(t)\Pi_1^3\beta - \mathcal{L}S_p(t)\alpha\|_{X_1^\varepsilon}^2 \leq \|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_1}^2 + L^2 C'_2 \varepsilon,$$

and (41) becomes

$$\begin{aligned} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon} &\leq C'_1 \varepsilon^{\gamma'} + \left( \|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_1}^2 + L^2 C'_2 \varepsilon \right)^{1/2} \\ &\leq C'_1 \varepsilon^{\gamma'} + \|S_p(t)\Pi_1^3\beta - S_p(t)\alpha\|_{Y_1} + L C'_2 \sqrt{\varepsilon} \\ &\leq C'_3 \varepsilon^{\gamma'_0} + \|S_p(t)\alpha - S_p(t)\Pi_1^3\beta\|_{Y_1} \end{aligned}$$

where  $C'_3 := 2 \max\{C'_1, L C'_2\}$  and  $\gamma'_0 := \min\{\frac{1}{2}, \gamma'\}$ , and in turn,

$$\begin{aligned} (42) \quad &\sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|\mathcal{L}S_p(t)\alpha - S_\varepsilon(t)\beta\|_{X_1^\varepsilon} \\ &\leq C'_3 \varepsilon^{\gamma'_0} + \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3\beta\|_{Y_1}. \end{aligned}$$

Recall that by using the compatibility criterion #4,

$$\Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) = \bigcap_{s \geq c_2^p} \bigcup_{t \geq s} K_p(t) \left( \bigcup_{r \geq c_1^p} K_p(r) \Pi_1^1 \mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon \right),$$

and any  $\Pi_1^3\beta \in \Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$  can be written as  $\Pi_1^3\beta = K_p(t_0)K_p(r_0)\Pi_1^1(\xi + m_\varepsilon(\xi))$  for some  $t_0 \geq c_2^p$ ,  $r_0 \geq c_1^p$ , and  $\xi \in \mathcal{W}^\varepsilon$ . Again, by the compatibility criterion #2,  $\mathcal{W}^\varepsilon = \mathcal{W}^p \times \mathcal{W}^p$ , so we can write  $\xi = (\chi, \psi)$  for some  $\chi, \psi \in \mathcal{W}^p$ . Also, by lemma 4.4,

$$\xi + m_\varepsilon(\xi) = (\chi, \psi) + m_\varepsilon(\chi, \psi) = (\chi, \psi) + (f(\chi), g(\psi)) = (\chi + f(\chi), \psi + g(\psi))$$

for some  $f \in \tilde{\mathcal{G}}_L^0$  and  $g \in \mathcal{G}_L^0$ . Consequently,

$$\begin{aligned} S_p(t)\Pi_1^3\beta &= S_p(t)K_p(t_0)K_p(r_0)\Pi_1^1(\chi + f(\chi), \psi + g(\psi)) \\ &= S_p(t)K_p(t_0)K_p(r_0)(\chi + f(\chi)). \end{aligned}$$

In a similar fashion, we can write

$$S_p(t)\alpha = S_p(t)K_p(t_1)K_p(r_1)K_p(s_1)(\zeta + m_p(\zeta))$$

for some  $t_1 \geq c_2^p$ ,  $r_1 \geq c_1^p$ ,  $s_1 \geq c_0^p$ , and  $\zeta \in \mathcal{W}^p$ . Using hypothesis 4.5, we have,

$$\begin{aligned} &\|S_p(t)\alpha - S_p(t)\Pi_1^3\beta\|_{Y_1} \\ &= \|S_p(t)K_p(t_1)K_p(r_1)K_p(s_1)(\zeta + m_p(\zeta)) - S_p(t)K_p(t_0)K_p(r_0)(\chi + f(\chi))\|_{Y_1} \\ &= \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_0)(\chi + f(\chi))\|_{Y_1} \end{aligned}$$

where  $\sigma_1 := t + t_1 + r_1 + s_1 \geq \tau_3^p + c_2^p + c_1^p + c_0^p$  and  $\sigma_0 := t + t_0 + r_0 \geq \tau_3^p + c_2^p + c_1^p$ .

Fix  $\sigma_0 \geq \tau_3^p + c_2^p + c_1^p$  and  $\chi \in \mathcal{W}^p$  so that  $\Pi_1^3 \beta = S_p(\sigma_0)(\chi + f(\chi)) \in \Pi_1^3 \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)$ . Then

$$\begin{aligned} & \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \\ & \leq \inf_{\substack{\sigma_0 \geq \tau_3^p + c_2^p + c_1^p \\ \chi \in \mathcal{W}^p}} \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_0)(\chi + f(\chi))\|_{Y_1} \\ & \leq \inf_{\chi \in \mathcal{W}^p} \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_0)(\chi + f(\chi))\|_{Y_1} \end{aligned}$$

for all  $\sigma_0 \geq \tau_3^p + c_2^p + c_1^p$ ; in particular, we can choose  $\sigma_0 = \sigma_1$ ; hence,

$$\inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \leq \inf_{\chi \in \mathcal{W}^p} \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_1)(\chi + f(\chi))\|_{Y_1}$$

and

$$\begin{aligned} & \sup_{\zeta \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \\ & \leq \sup_{\zeta \in \mathcal{W}^p} \inf_{\chi \in \mathcal{W}^p} \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_1)(\chi + f(\chi))\|_{Y_1} \\ & \leq \sup_{\zeta \in \mathcal{W}^p} \|S_p(\sigma_1)(\zeta + m_p(\zeta)) - S_p(\sigma_1)(\chi + f(\chi))\|_{Y_1} \end{aligned}$$

for all  $\chi \in \mathcal{W}^p$ ; in particular, we can choose  $\chi = \zeta$ ; so,

$$\sup_{\zeta \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \leq \sup_{\zeta \in \mathcal{W}^p} \|S_p(\sigma_1)m_p(\zeta) - S_p(\sigma_1)f(\zeta)\|_{Y_1}.$$

Applying (31) of hypothesis 4.6 yields,

$$\begin{aligned} (43) \quad & \sup_{\zeta \in \mathcal{W}^p} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \\ & \leq \sup_{\zeta \in \mathcal{W}^p} \|S_p(\sigma_1)m_p(\zeta) - S_p(\sigma_1)f(\zeta)\|_{Y_1} \leq C'_4 \varepsilon^{\lambda'}. \end{aligned}$$

Collecting (39), (40), (42), and (43), yields,

$$\partial_{X_1^\varepsilon} (\mathcal{L}\widehat{\mathcal{M}}_3^0, \widehat{\mathcal{M}}_3^\varepsilon) \leq C'_3 \varepsilon^{\gamma'_0} + C'_4 \varepsilon^{\lambda'},$$

and therefore,

$$(44) \quad \partial_{X_1^\varepsilon} (\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq \Lambda_2 \varepsilon^{\phi_2}$$

where  $\Lambda_2 := 2 \max\{C'_3, C'_4\}$  and  $\phi_2 := \min\{\gamma'_0, \lambda'\}$ . Furthermore, combining (38) and (44) we conclude

$$\text{dist}_{X_1^\varepsilon} (\mathbb{M}_3^\varepsilon, \mathbb{M}_3^0) \leq \Lambda \varepsilon^\phi$$

where  $\Lambda := \max\{\Lambda_1, \Lambda_2\}$  and  $\phi := \min\{\phi_1, \phi_2\}$ , and therefore the family  $(\mathbb{M}_3^\varepsilon)_{\varepsilon \in [0,1]}$  is robust at  $\varepsilon_0 = 0$  in the topology of  $X_1^\varepsilon$ .  $\square$

At this time we cannot show hypothesis 4.6 also holds for the model problem; however, we are able to show that a different version of hypothesis 4.6 holds to give us a weaker result.

## 5. THE MODEL PROBLEM

In this section we show the hypotheses 2.1, 2.2, 3.2, 3.5, 3.6, 3.8, 3.10, 4.1, 4.5, and a weaker version of 4.6 hold for a dissipative evolution equation under a singular perturbation of hyperbolic relaxation type.

**5.1. The framework.** We summarize some important facts about the Laplace operator here. These facts are crucial for the framework of both the parabolic and the hyperbolic problems. Let  $\Omega = (0, \pi) \subset \mathbb{R}^1$ . Let  $A := -\Delta$  where

$$D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$$

denotes the domain of the Laplacian in  $L^2(0, \pi)$  with Dirichlet boundary conditions on  $\Omega$  (and from now on we will omit the domain  $\Omega = (0, \pi)$  from the spaces; e.g.  $L^2(0, \pi) =: L^2$ ).

- (1) The operator  $-A = \Delta$  generates a  $C^0$ -semigroup (in fact, an *analytic* semigroup) on  $L^2$  (cf. e.g. [4, §7.4.3], [23, §9.2, Ex. 4], or [27, Chap. 7, Thm. 2.7]).
- (2) Recall  $L^2 \cong (L^2)'$ . Also,

$$H_0^1 \hookrightarrow L^2 \cong (L^2)' \hookrightarrow H^{-1},$$

densely and continuously, with compact injection  $H_0^1 \hookrightarrow L^2$ . (Concerning the density of the imbedding  $H_0^1 \hookrightarrow L^2$ , see [31, §II.1.2].)

- (3)  $A = -\Delta \in \mathcal{L}(H_0^1, H^{-1})$  is self-adjoint and strictly positive (cf. [24, §A.5.5]). Consequently, the operator  $A^{-1} = (-\Delta)^{-1} \in \mathcal{L}(L^2, L^2)$  is compact and the operator  $A = -\Delta$  admits a countable set of real eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  with a corresponding orthonormal system of eigenfunctions  $\{\omega_n\}_{n=1}^\infty$  spanning  $L^2$ .

Now we can define, for  $s \geq 0$ ,  $H_s := D(A^{s/2})$ . Recall, the imbedding

$$H_r \hookrightarrow H_s$$

is continuous and compact whenever  $r > s \geq 0$  (cf. [24, §A.5.5 & Thm. A69]).

The problem under consideration is an autonomous (weakly) damped Klein-Gordon wave equation with a singular hyperbolic perturbation; or in other words, we consider the Chafee-Infante reaction diffusion equation with a hyperbolic relaxation

$$(45) \quad \varepsilon u_{tt} + u_t - \Delta u + u^3 - u = f \text{ in } Q,$$

for  $\varepsilon \in [0, 1]$  and  $f \in L^2$ , with Dirichlet boundary conditions

$$(46) \quad u(t, 0) = u(t, \pi) = 0,$$

and initial conditions

$$(47) \quad u(0, \cdot) = u_0, \quad \varepsilon u_t(0, \cdot) = \varepsilon u_1.$$

We set  $Y_i = H_i$ , for  $i = 0, 1, 2, 3$ , and let

$$X_k = H_k \times H_{k-1}, \quad k = 1, 2, 3,$$

be endowed with the norm whose square is given by, for  $(u, v) \in X_k$ ,

$$\|(u, v)\|_{X_k}^2 = \|u\|_k^2 + \|v\|_{k-1}^2 = \|A^{k/2}u\|^2 + \|A^{(k-1)/2}v\|^2$$

Here,  $\|\cdot\|_k$  denotes the norm in  $H_k$ ,  $\|\cdot\| = \|\cdot\|_0$  denotes the norm in  $L^2$ ,  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$  product, and we denote by  $|\cdot|_p$  the norm in  $L^p$ ,  $1 \leq p \leq \infty$ . For  $\varepsilon \in (0, 1]$ , consider in  $X_k$  the equivalent  $\varepsilon$ -weighted norm whose square is given by

$$\|(u, v)\|_{X_k^\varepsilon}^2 := \|u\|_k^2 + \varepsilon \|v\|_{k-1}^2 = \|A^{k/2}u\|^2 + \varepsilon \|A^{(k-1)/2}v\|^2$$

and call  $X_k^\varepsilon$  the corresponding normed space. Recall, when  $\varepsilon = 0$ , the space  $X_k^0$  reduces to the first component  $H_k = D(A^{k/2})$ .

Before we establish the hypotheses for both the parabolic problem and the hyperbolic problem, observe that the spaces  $X_k^0$  denote a regularized energetic phase space for the parabolic problem; e.g.,  $X_1^0 = H_0^1$  rather than  $X_p^0 = L^2$ , the natural setting for the parabolic problem. For the construction of the robust family of compact inertial manifolds, our base phase space will be  $X_1^\varepsilon = H_0^1 \times L^2$  for  $\varepsilon \in [0, 1]$ , so an extra regularity result is expected from the parabolic problem. However, for the hyperbolic problem, the product structure of  $X_1^\varepsilon$  is natural and no further difficulties from our choice of the base energetic phase space should result.

**5.2. The parabolic problem.** In the parabolic case, problem (45)-(47) can be written as the abstract Cauchy problem:

$$(48) \quad \frac{du(t)}{dt} + A_0 u(t) = F_0(u(t)), \quad t > 0$$

with the initial condition

$$(49) \quad u(0) = u_0,$$

where  $A_0 = -\Delta$ ,  $F_0(u) = f - u^3 + u$ , and  $u_0, f \in L^2$ .

The spaces defined above by  $Y_i = H_i$ ,  $i = 0, 1, 2, 3$ , satisfy hypothesis 3.5. We will show that  $A_0 = -\Delta$  satisfies the spectral gap condition relative to the Lipschitz constant of  $F_0$  below.

We know that the map  $g$  given by  $g(u) = u^3 - u$  is locally Lipschitz continuous from  $H_0^1$  into  $L^2$ , so it follows that  $F_0(u) = f - g(u)$  is as well. Indeed, given  $M > 0$ , and  $u, v \in H_0^1$  with  $\|u\|_1 \leq M$  and  $\|v\|_1 \leq M$ , then

$$\|F_0(u) - F_0(v)\| \leq |(u^2 + uv + v^2 - 1)|_\infty \|(u - v)\| \leq CM\|u - v\|_1,$$

where the constant  $C > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ . But we are required to meet the stronger condition that  $F_0 \in C_b(L^2, L^2)$  satisfying the global Lipschitz condition (3) for  $\ell_{F_0} > 0$ , depending on  $F_0$  but independent of  $x, y \in L^2$ . To obtain this, we modify  $F_0$  outside an absorbing set of  $L^2$  (which is yet to be determined) making  $F_0$  both globally bounded and Lipschitz continuous from  $L^2$  into  $L^2$ . When this is at hand, we borrow [24, §5.7.1] where it is shown that  $A_0 = -\Delta$  satisfies the spectral gap condition relative to the modified nonlinearity  $\tilde{F}_0$  (defined below).

To modify the nonlinear term  $F_0$ , fix  $\delta > 1$  and choose a function  $\gamma \in C_b^\infty(\mathbb{R})$  such that

$$|\gamma(r)| \leq 2\delta - 1 \text{ and } |\gamma'(r)| \leq 1 \quad \forall r \in \mathbb{R},$$

and

$$\gamma(r) = r \text{ when } |r| \leq \delta.$$

Then we set

$$\tilde{F}_0(u) := f - (\gamma(u))^3 + \gamma(u).$$

Following [24, Prop. 5.51], we claim the following (the authors assume  $f \equiv 0$ ; however, the same claim holds).

*Lemma 5.1.* For any  $\delta > 1$ ,  $\tilde{F}_0$  is globally bounded and Lipschitz continuous from  $L^2$  into  $L^2$  with Lipschitz constant

$$(50) \quad \ell_{\tilde{F}_0} := 1 + 3(2\delta - 1)^2.$$

*Proof.* The boundedness of  $\tilde{F}_0$  in  $L^2$  follows from the boundedness of  $\gamma$  on  $\mathbb{R}$ ; indeed,

$$\|\tilde{F}_0(u)\| \leq \|f\| + |\gamma(u)|_6^3 + \|\gamma(u)\| \leq \|f\| + (2\delta - 1)\sqrt{\pi}(1 + (2\delta - 1)^2).$$

For the Lipschitz continuity of  $\tilde{F}_0$  from  $L^2$  into  $L^2$ ,

$$\begin{aligned} \|\tilde{F}_0(u) - \tilde{F}_0(v)\|^2 &= \int_0^\pi |(\gamma(u))^2 + \gamma(u)\gamma(v) + (\gamma(v))^2 - 1|^2 |\gamma(u) - \gamma(v)|^2 \, dx \\ &\leq (3(2\delta - 1)^2 + 1)^2 \|u - v\|^2. \end{aligned}$$

This proves the claim.  $\square$

Lemma 5.1 establishes hypothesis 2.1. We are now in a position to show that the spectral gap property holds for the operator  $A_0 = -\Delta$  with respect to the modified nonlinearity  $\tilde{F}_0$ . We need to find  $N_p \in \mathbb{N}$  such that the eigenvalues of  $A_0 = -\Delta$  in  $L^2$  with domain  $H^2 \cap H_0^1$  satisfy (5). Recall  $\Omega = (0, \pi)$ , so the eigenvalues of  $A_0 = -\frac{d^2}{dx^2}$  are  $\lambda_n = n^2$ ,  $n = 1, 2, 3, \dots$  (cf. e.g. [2, Ex. 2.19]). Thus condition (5) reads

$$(N_p + 1)^2 - (N_p)^2 > 4(1 + 3(2\delta - 1)^2),$$

and hence, is satisfied if and only if

$$(51) \quad N_p > 24\delta(\delta - 1) + \frac{15}{2}.$$

Denote by  $N_{p*}$  the minimum integer  $N_p$  satisfying (51).

This establishes hypothesis 2.2 (with parts 3 & 4). Consequently, for each  $u_0, f \in L^2$ , there is a unique mild solution to the IVP (48)-(49) satisfying, for all  $t \in [0, T]$ ,

$$(52) \quad u(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-\tau)}(f - (u(\tau))^3 + u(\tau)) \, d\tau.$$

We do not yet define the Lipschitz continuous parabolic semiflow  $S_p(t) : L^2 \rightarrow L^2$  through the mild solution since, as we will see below, such solutions are also weak solutions. Proceeding, given  $\delta > 1$  we can determine  $\ell_{\tilde{F}_0}$  as in (50) above and choose integer  $N_p$  satisfying (51). With these we set  $\mathfrak{I}_1 := \text{span}\{\omega_n(x)\}_{n=1}^{N_p}$ . We know  $\omega_n(x) = \sin(nx)$ , for  $n = 1, 2, 3, \dots$ , are the eigenvectors of  $A_0 = -\frac{d^2}{dx^2}$  in  $L^2$  with domain  $H^2 \cap H_0^1$  (these are the eigenvectors corresponding to the eigenvalues  $\lambda_n = n^2$  mentioned above). Also set  $\mathfrak{I}_2 := \mathfrak{I}_1^\perp$ . Then, by [24, Thm. 5.42] for example, there exists a map

$$m_p \in \mathcal{G}_{\ell_{\tilde{F}_0}} := \{\Psi \in C_b(\mathfrak{I}_1; \mathfrak{I}_2) : \|\Psi(\chi) - \Psi(\psi)\| \leq \ell_{\tilde{F}_0}\|\chi - \psi\| \ \forall \chi, \psi \in \mathfrak{I}_1\}$$

such that the subset of  $L^2$  defined by the graph of  $m_p$  on  $\mathfrak{I}_1$ ,

$$(53) \quad \mathcal{M}_p^0 := \text{graph}(m_p) = \{\xi + m_p(\xi) : \xi \in \mathfrak{I}_1\},$$

is an inertial manifold for  $S_p$  (defined below) in  $L^2$ .

Let  $T > 0$ ,  $u_0 \in L^2$ , and  $f \in H^{-1}$ . A function  $u$  is called a *weak solution* to the parabolic problem,

$$(54) \quad \begin{cases} u_t - \Delta u + u^3 - u = f & \text{in } (0, \pi) \times (0, T) =: Q_T \\ u(t, 0) = u(t, \pi) = 0 \\ u(0, \cdot) = u_0, \end{cases}$$

on the interval  $[0, T]$  if

$$(55) \quad u \in C([0, T]; L^2) \cap L^2(0, T; H_0^1) \cap L^4(Q_T),$$

and for all

$$\varphi \in \mathcal{P}(0, T) := \{h \in L^2(0, T; H_0^1) : h_t \in L^2(0, T; H^{-1}), h(T, \cdot) = 0\} \cap L^4(Q_T),$$

there holds

$$(56) \quad \begin{aligned} & \int_0^T -\langle u, \varphi_t \rangle + \langle \nabla u, \nabla \varphi \rangle + \langle u^3 - u, \varphi \rangle dt \\ &= \int_0^T (f, \varphi)_{H^{-1} \times H_0^1} dt + \langle u_0, \varphi(0) \rangle. \end{aligned}$$

(Again, the term  $(f, \varphi)_{H^{-1} \times H_0^1}$  denotes the dual pair of  $f \in H^{-1}$  and  $\varphi \in H_0^1$  defined by, through the Riesz representation theorem,  $(f, \varphi)_{H^{-1} \times H_0^1} = \int_{\Omega} f \varphi \, dx$ .)

It is well-known that for all  $T > 0$ ,  $u_0 \in L^2$ , and  $f \in H^{-1}$ , there exists a unique weak solution  $u$  of the parabolic problem (54). Moreover, the solution depends continuously on the initial data  $u_0$ , and the function  $t \mapsto \|u(t)\|$  is absolutely continuous on  $[0, T]$  (cf. [24, Prop. 3.7 & Thm. 3.9]). Concerning the regularity of the weak solution, when  $u_0 \in H_0^1$  and  $f \in L^2$ , the weak solution to the parabolic problem satisfies

$$(57) \quad u \in C([0, T]; H_0^1) \cap L^2(0, T; H^2), \quad u_t \in L^2(0, T; L^2).$$

Moreover, the function  $t \mapsto \|\nabla u(t)\|$  is absolutely continuous on  $[0, T]$  (cf. [24, Thm. 3.9]). Hence, the maps  $t \mapsto \|u(t)\|$  and  $t \mapsto \|\nabla u(t)\|$  are differentiable almost everywhere (cf. e.g. [30, Chap. 5, Cor. 12]).

We should mention that the proof of the existence of weak solutions by the Galerkin method shows that a finite set of ODEs (obtained by “projecting” the infinite dimensional problem onto a finite dimensional subspace spanned by a finite set eigenvectors of the Laplacian) possesses a *maximal solution*; e.g., for  $t \in [0, t_{\max}]$ , for some  $t_{\max} \in (0, T]$ . Then with the use of an *a priori* estimate, we show that the approximate solutions cannot exhibit blow-up behavior as  $t \rightarrow t_{\max}$ . Thus,  $t_{\max} = T$ , and by the arbitrariness of  $T > 0$ , we may indeed investigate the “global behavior” of the weak solutions  $u(t)$  as  $t \rightarrow \infty$ .

We already have shown the existence of a mild solution given  $u_0, f \in L^2$ , and it is easy to see that the solution given by (52) is also a weak solution. Indeed, since  $f$  is assumed to be in  $L^2$  for the mild formulation, the dual pair is replaced with the  $L^2$  product  $\langle f, \varphi \rangle$ , and (56) holds if and only if for all  $\varphi \in \mathcal{P}(0, T)$ ,

$$\int_0^T \langle u_t - \Delta u + u^3 - u - f, \varphi \rangle dt = 0.$$

But  $u$  satisfies the PDE in (54) and hence the mild solution  $u$  is also a weak solution. In any case we define the Lipschitz continuous parabolic semiflow  $S_p(t) : L^2 \rightarrow L^2$  through the weak solution:

$$(58) \quad S_p(t)u_0(x) := u(t, x, u_0).$$

From (55) and (57), we infer that the semiflow  $S_p$  defined by (58) is continuous in  $L^2$ , and in  $H_0^1$ , leaving us to later verify hypothesis 3.8 for  $j = 1, 2$ .

Concerning hypothesis 3.2, we note that no decomposition is required in the parabolic case. Indeed, it is well-known that for  $u_0, f \in L^2$ , and for all  $t \geq \ln 2$ ,

$$(59) \quad \|u(t)\|_1^2 \leq \frac{Ce^t}{e^t - 1} \leq 2C,$$

where here  $C > 0$  can be explicitly computed to be  $C = 2(2\|f\|^2 + \|u_0\|^2 + 3\sqrt{\pi})$  (cf. [24, §3.3.3]). So in this case we set (trivially)  $Z_p = 0$  and  $K_p = S_p$ , and deduce hypotheses 3.2 and also 4.5; in anticipation of hypothesis 3.6, we set  $c_0^p = \ln 2$ . For hypothesis 3.6, it remains to determine  $c_1^p$  and  $c_2^p$ . We will show hypotheses 3.6 and 3.8 for  $j = 1, 2$ , as well as hypothesis 3.10 hold for the parabolic problem using standard regularity methods.

*Lemma 5.2.* Let  $u_0, f \in L^2$ . There is  $C > 0$  such that for all  $t \geq \ln 2$ ,

$$\int_{\ln 2}^t \|\Delta u(\tau)\|^2 d\tau \leq C + (\|f\|^2 + C)t.$$

*Proof.* Multiplication of the PDE in (54) in  $L^2$  by  $-2\Delta u(t)$  yields, for almost all  $t \geq 0$  (neglecting the argument  $t$ ),

$$(60) \quad \frac{d}{dt} \|\nabla u\|^2 + 2\|\Delta u\|^2 + 6\langle u^2 \nabla u, \nabla u \rangle = 2\langle f + u, -\Delta u \rangle.$$

Estimating the right hand side for  $t \geq \ln 2$  gives,

$$2\langle f + u, -\Delta u \rangle \leq \|f\|^2 + \|\Delta u\|^2 + 2\|\nabla u\|^2 \leq \|f\|^2 + \|\Delta u\|^2 + C,$$

where  $C > 0$  depends on the uniform bound with respect to  $t \geq \ln 2$  on the solution  $u(t)$  in  $H_0^1$  (see (59)) (also, this is why we need  $u_0 \in L^2$ ). Thus, omitting the positive term  $6\langle u^2 \nabla u, \nabla u \rangle$  in (60) and applying the above estimate, we have for almost all  $t \geq \ln 2$ ,

$$(61) \quad \frac{d}{dt} \|\nabla u\|^2 + \|\Delta u\|^2 \leq \|f\|^2 + C.$$

Integrating (61) yields, for all  $t \geq \ln 2$ ,

$$\|\nabla u(t)\|^2 + \int_{\ln 2}^t \|\Delta u(\tau)\|^2 d\tau \leq \|\nabla u(\ln 2)\|^2 + (\|f\|^2 + C)t.$$

Since  $\|\nabla u(\ln 2)\| < \infty$ , the claim now follows.  $\square$

*Lemma 5.3.* Let  $u_0 \in L^2$  and  $f \in H^1$ . Then  $u$  is uniformly bounded with respect to  $t$  from  $[\ln 3, \infty)$  into  $H^2$ .

*Proof.* We formally multiply the PDE in (54) by  $2(e^t - 2)(-\Delta)^2 u(t)$  in  $L^2$  to obtain (temporarily suspending the argument  $t$  from  $u$  again),

$$(62) \quad \begin{aligned} \frac{d}{dt} ((e^t - 2)\|\Delta u\|^2) + 2(e^t - 2)\|\nabla \Delta u\|^2 \\ = e^t \|\Delta u\|^2 + 2(e^t - 2)\langle f - u^3 + u, (-\Delta)^2 u \rangle \\ = \|\Delta u\|^2 + (e^t - 2)/2 \langle f - 2u^3 + 3u, (-\Delta)^2 u \rangle. \end{aligned}$$

The multiplication above is only formal since we do not know that  $\nabla \Delta u(t) \in L^2$  when  $u_0 \in L^2$  and  $f \in H^1$ ; moreover,  $-\Delta u$  does not satisfy the Dirichlet boundary conditions; i.e., we do not yet know that  $\|\nabla \Delta u\| < \infty$  or

$$\langle -\Delta u, (-\Delta)^2 u \rangle = \langle \nabla \Delta u, \nabla \Delta u \rangle.$$

To justify (62), the following estimates can be verified for the Galerkin approximations of  $u$  which would be used to determine the existence of a weak solution; then one argues that the same estimate holds in the limit for the solution  $u$ . To obtain

$\Delta u|_{\partial\Omega} = 0$ , we need to use the Galerkin basis given by the eigenfunctions of the Laplacian with Dirichlet boundary boundary conditions; i.e., for  $n \in \mathbb{N}$ ,

$$\begin{cases} -\Delta\omega_n = \lambda_n\omega_n \\ \omega_n|_{\partial\Omega} = 0, \end{cases}$$

then  $\Delta\omega_n|_{\partial\Omega} = 0$ . The reader is referred to, for example, [24, Thm. 3.9] or [31, Chap. III, Thm. 1.1] for further details.

Using the fact that  $f \in H^1$  and a weighted Cauchy inequality, the product on the right hand side of (62) can be controlled by

$$\langle 2f - 2u^3 + 3u, (-\Delta)^2 u \rangle \leq \frac{11}{4}\|\nabla f\|^2 + 2\|\nabla\Delta u\|^2 + \frac{33}{4}\|u^2\nabla u\|^2 + \frac{33}{8}\|\nabla u\|^2.$$

Recall that  $H_0^1 \hookrightarrow L^\infty$  since  $N = 1$ , and recall that  $m(\Omega) = \pi$ ; also, since  $u(t)$  is uniformly bounded with respect to  $t \geq \ln 2$  in  $H_0^1$ , there is  $C > 0$  such that

$$(63) \quad \frac{33}{4}\|u^2\nabla u\|^2 + \frac{33}{8}\|\nabla u\|^2 \leq \frac{33}{4}\pi|u|_\infty^4\|\nabla u\|^2 + \frac{33}{8}\|\nabla u\|^2 \leq C.$$

Then (62) becomes

$$(64) \quad \frac{d}{dt}((e^t - 2)\|\Delta u\|^2) \leq \|\Delta u\|^2 + e^t \left( \frac{11}{4}\|\nabla f\|^2 + C \right).$$

With the aid of lemma 5.2, integration of (64) yields for all  $t \geq \ln 2$ ,

$$(e^t - 2)\|\Delta u(t)\|^2 \leq \int_{\ln 2}^t \|\Delta u(\tau)\|^2 d\tau + Ce^t \leq C + Ct + Ce^t \leq Ce^t,$$

where now  $C$  denotes a sufficiently large generic constant, now also depending on  $f$ , but independent of  $t$ . Therefore, for all  $t > \ln 2$ ,

$$\|\Delta u(t)\|^2 \leq \frac{Ce^t}{e^t - 2},$$

and the claim follows.  $\square$

*Lemma 5.4.* Let  $u_0 \in L^2$  and  $f \in H^1$ . There is  $C > 0$  independent of  $t$  such that for all  $t \geq \ln 3$ ,

$$\int_{\ln 3}^t \|\nabla\Delta u(\tau)\|^2 d\tau \leq C + (3\|\nabla f\|^2 + C)t.$$

*Proof.* Multiply the PDE in (54) in  $L^2$  by  $2(-\Delta)^2 u(t)$ , we obtain for almost all  $t \geq 0$

$$\frac{d}{dt}\|\Delta u\|^2 + 2\|\nabla\Delta u\|^2 = 2\langle f - u^3 + u, (-\Delta)^2 u \rangle.$$

The map  $t \mapsto \|\Delta u(t)\|$  is absolutely continuous (cf. [31, Chap. II, Lem. 3.2] and also [32, Chap. 3, Lem. 1.2] for more details). The right hand side above is controlled by

$$3\|\nabla f\|^2 + \|\nabla\Delta u\|^2 + C,$$

where  $C > 0$  is similar to the one found in (63) but now follows from lemma 5.3. This gives

$$\frac{d}{dt}\|\Delta u\|^2 + \|\nabla\Delta u\|^2 \leq 3\|\nabla f\|^2 + C,$$

and, integrating over  $[\ln 3, \infty)$ ,

$$\|\Delta u(t)\|^2 + \int_{\ln 3}^t \|\nabla\Delta u(\tau)\|^2 d\tau \leq \|\Delta u(\ln 3)\|^2 + (3\|\nabla f\|^2 + C)t,$$

from which the claim follows.  $\square$

*Lemma 5.5.* Let  $u_0 \in L^2$  and  $f \in H^2$ . Then  $u$  is uniformly bounded with respect to  $t$  from  $[\ln 4, \infty)$  into  $H^3$ .

*Proof.* Formally multiply the PDE in (54) by  $2(e^t - 3)(-\Delta)^3 u(t)$  in  $L^2$  to obtain,

$$\begin{aligned} (65) \quad & \frac{d}{dt} ((e^t - 3)\|\nabla \Delta u\|^2) + 2(e^t - 3)\|\Delta^2 u\|^2 \\ &= e^t \|\nabla \Delta u\|^2 + 2(e^t - 3)\langle f - u^3 + u, (-\Delta)^3 u \rangle \\ &= \|\nabla \Delta u\|^2 + (e^t - 3)\langle 2f - 2u^3 + 3u, (-\Delta)^3 u \rangle. \end{aligned}$$

Using the facts that  $\Omega$  is bounded,  $H^2 \cap H_0^1 \hookrightarrow H_0^1 \hookrightarrow L^\infty$ , and now that  $u$  is uniformly bounded with respect to  $t$  in  $H^2 \cap H_0^1$  for  $t \geq \ln 3$  thanks to lemma 5.3, there is a constant  $C > 0$ , uniform in  $t$ , such that

$$\langle 2f - 2u^3 + 3u, (-\Delta)^3 u \rangle \leq \frac{23}{4}\|\Delta f\|^2 + 2\|\Delta^2 u\|^2 + C.$$

Then (65) becomes

$$(66) \quad \frac{d}{dt} ((e^t - 3)\|\nabla \Delta u\|^2) \leq \|\nabla \Delta u\|^2 + e^t \left( \frac{23}{4}\|\Delta f\|^2 + C \right).$$

With the aid of lemma 5.4, integration of (66) yields for all  $t \geq \ln 3$ ,

$$(e^t - 3)\|\nabla \Delta u(t)\|^2 \leq \int_{\ln 3}^t \|\nabla \Delta u(\tau)\|^2 d\tau + Ce^t \leq C + Ct + Ce^t \leq Ce^t,$$

where now  $C > 0$  denotes a sufficiently large generic constant independent of  $t$ . Therefore, for all  $t > \ln 3$ ,

$$\|\nabla \Delta u(t)\|^2 \leq \frac{Ce^t}{e^t - 3},$$

and the claim follows.  $\square$

*Remark 5.6.* From [24, §3.3.3] and the conclusions of the proofs of lemmas 5.3 and 5.5, observe that the estimates

$$\|u(t)\|_{j+1}^2 \leq \frac{Ce^t}{e^t - (j+1)} \leq 2C, \quad j = 0, 1, 2,$$

hold whenever  $t \geq \ln(j+2)$ . This in fact establishes hypothesis 3.6 with  $c_j^p := \ln(j+2)$ . Observe that this choice is not sharp; of course our choice is dictated by definiteness and aesthetics. Below we give further estimates which motivate the justification of hypotheses 3.8 (for  $j = 1, 2$ ) and 3.10 for the parabolic problem. Indeed, the follow lemmas are used to provide the existence of a positively invariant absorbing set in the spaces  $L^2$ ,  $H_0^1$ , and  $H^2 \cap H_0^1$ .

The following lemma applies to a general case but will come in to use immediately.

*Lemma 5.7.* Let  $f \in AC([a, b])$  and  $\alpha, \beta \in L^1(a, b)$ . If for almost all  $t \in [a, b]$

$$\frac{df}{dt} \leq \alpha(t) + \beta(t)f(t),$$

then

$$f(t) \leq f(a) \exp \left( \int_a^t \beta(s) ds \right) + \int_a^t \exp \left( \int_\tau^t \beta(s) ds \right) \alpha(\tau) d\tau \quad \forall t \in [a, b].$$

In particular, if  $\alpha$  and  $\beta$  are constants with  $\beta \neq 0$ , then

$$f(t) \leq e^{\beta t} f(0) + \frac{\alpha}{\beta} (e^{\beta t} - 1) \quad \forall t \in [0, T].$$

*Proof.* Cf. [24, Proposition 2.63]. □

**Lemma 5.8.** Let  $u_0, f \in L^2$ . Then for all  $t \geq 0$ ,

$$\|u(t)\|^2 \leq e^{-2t} \|u_0\|^2 + \frac{1}{2} (3\sqrt{\pi} + \|f\|^2).$$

*Proof.* The proof is taken from [24, Prop. 3.12]. Multiply the PDE in (54) by  $2u(t)$  to obtain, for almost all  $t \geq 0$  (neglecting the argument  $t$ ),

$$(67) \quad \frac{d}{dt} \|u\|^2 + 2\|\nabla u\|^2 + 2|u|_4^4 = 2\langle f + u, u \rangle \leq 3\sqrt{\pi} + \|f\|^2 + |u|_4^4.$$

Recalling the Poincaré inequality on  $\Omega = (0, \pi)$ , then  $2\|u\|^2 \leq 2\|\nabla u\|^2$ , and

$$\frac{d}{dt} \|u\|^2 + 2\|u\|^2 \leq 3\sqrt{\pi} + \|f\|^2.$$

The claim follows from the linear differential inequality (Lemma 5.2). □

**Lemma 5.9.** Let  $u_0 \in H_0^1$  and  $f \in L^2$ . Then for all  $t \geq 0$ ,

$$(68) \quad \|\nabla u(t)\|^2 \leq e^{-t/2} (\|u_0\|^2 + \|\nabla u_0\|^2) + 2(3\sqrt{\pi} + 2\|f\|^2).$$

*Proof.* The proof is similar to [24, Prop. 4.6]. Multiplication of the PDE in (54) in  $L^2$  by  $-2\Delta u(t)$  yields, for almost all  $t \geq 0$  (neglecting the argument  $t$ ),

$$\frac{d}{dt} \|\nabla u\|^2 + 2\|\Delta u\|^2 + 6\langle u^2 \nabla u, \nabla u \rangle = 2\langle f + u, -\Delta u \rangle.$$

Estimating the right hand side gives,

$$2\langle f + u, -\Delta u \rangle \leq \|f\|^2 + \|\Delta u\|^2 + 2\|\nabla u\|^2.$$

Thus, omitting the positive term  $6\langle u^2 \nabla u, \nabla u \rangle$  in (60) and applying the above estimate, we have, for almost all  $t \geq 0$ ,

$$\frac{d}{dt} \|\nabla u\|^2 + \|\Delta u\|^2 \leq \|f\|^2 + 2\|\nabla u\|^2.$$

Adding in (67) to (61) yields,

$$(69) \quad \frac{d}{dt} (\|u\|^2 + \|\nabla u\|^2) + \|\Delta u\|^2 \leq 3\sqrt{\pi} + 2\|f\|^2.$$

Using the Poincaré type estimate,

$$\frac{1}{2} (\|u\|^2 + \|\nabla u\|^2) \leq \|\Delta u\|^2,$$

then (69) becomes, for almost all  $t \geq 0$ ,

$$\frac{d}{dt} (\|u\|^2 + \|\nabla u\|^2) + \frac{1}{2} (\|u\|^2 + \|\nabla u\|^2) \leq 3\sqrt{\pi} + 2\|f\|^2.$$

Now using the linear differential inequality (Lemma 5.2) we have that for all  $t \geq 0$ ,

$$\|u(t)\|^2 + \|\nabla u(t)\|^2 \leq e^{-t/2} (\|u_0\|^2 + \|\nabla u_0\|^2) + 2(3\sqrt{\pi} + 2\|f\|^2).$$

The claim now follows. □

The following lemma establishes hypothesis 3.8 for the case  $j = 1$ .

*Lemma 5.10.* For any  $T > 0$ ,  $u_0 \in H^2 \cap H_0^1$ , and  $f \in H^1$ ,

$$u \in C([0, T]; H^2 \cap H_0^1).$$

*Proof.* We will begin by showing that  $u \in L^2(0, T; H^3)$ . Formally multiply the PDE in (54) by  $2(-\Delta)^2 u(t)$  in  $L^2$  to obtain, for almost all  $t \in [0, T]$ ,

$$(70) \quad \frac{d}{dt} \|\Delta u\|^2 + \|\nabla \Delta u\|^2 \leq \frac{15}{2} \|\nabla f\|^2 + C \|\nabla u\|^6 + \frac{15}{4} \|\nabla u\|^2,$$

where the constant  $C > 0$  is due to the imbedding  $H_0^1 \hookrightarrow L^\infty$ . As in lemma 5.3, the following estimates can be justified for the Galerkin approximations of  $u$  on the Galerkin basis given by the eigenfunctions of the Laplacian with Dirichlet boundary boundary conditions. By integrating (70) with respect to  $t$  on  $[0, T]$ , we have

$$\begin{aligned} \int_0^T \|\nabla \Delta u(t)\|^2 dt + \|\Delta u(t)\|^2 &\leq \|\Delta u_0\|^2 + \frac{15}{2} \|\nabla f\|^2 T + \\ &+ \int_0^T C(\|\nabla u(t)\|^6 + \|\nabla u(t)\|^2) dt. \end{aligned}$$

Integrating (68) over  $[0, T]$  produces,

$$\int_0^T \|\nabla u(t)\|^2 dt \leq CT,$$

where  $C > 0$  depends on  $u_0, f$  and  $\Omega$ . Moreover,

$$\int_0^T \|\nabla \Delta u(t)\|^2 dt \leq CT,$$

and  $u \in L^2(0, T; H^3)$ .

Once we also show that  $u_t \in L^2(0, T; H^1)$ , the claim follows from [21, §1.2, 1.3] (or see [24, Thm. A.80]). Formally multiply the equation (54) by  $-\Delta u_t(t)$  in  $L^2$  to produce

$$\|\nabla u_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 \leq 4 \|\nabla f\|^2 + \frac{1}{2} \|\nabla u_t\|^2 + 12\sqrt{\pi} |u|_\infty^2 \|\nabla u\|^2,$$

which is

$$\frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 \leq 4 \|\nabla f\|^2 + C$$

where  $C > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ . Integrating with respect to  $t$  on  $[0, T]$  yields

$$\int_0^T \|\nabla u_t(\tau)\|^2 d\tau \leq \|\Delta u_0\|^2 + 8 \|\nabla f\|^2 T + CT.$$

Thus,  $u_t \in L^2(0, T; H^1)$  and the claim follows.  $\square$

Now we show hypothesis 3.10 before finishing the justification of hypothesis 3.8. To show hypothesis 3.10, we construct an *a priori* estimate from which we deduce the existence of a bounded absorbing positively invariant closed ball

$$G_3^0 = B_3^0(0, \rho_3^p) := \{x \in H^3 \cap H_0^1 : \|x\|_3 \leq \rho_3^p\}.$$

*Lemma 5.11.* For all  $u_0 \in H^3 \cap H_0^1$  and  $f \in H^1$ , there exists  $\rho_3^p > 0$  such that the ball  $B_3^0(0, \rho_3^p)$  in  $H^3 \cap H_0^1$  is absorbing and positively invariant for  $S_p$  in  $H^3 \cap H_0^1$ .

*Proof.* Formally multiply the PDE in (54) by  $2(-\Delta)^3 u(t)$  in  $L^2$  to obtain, for almost all  $t \geq 0$ ,

$$(71) \quad \begin{aligned} \frac{d}{dt} \|\nabla \Delta u\|^2 + 2\|\Delta^2 u\|^2 &\leq 4\|\Delta f\|^2 + \|\Delta^2 u\|^2 + 8\|\Delta u\|^2 + 24|u|_\infty^2 |\nabla u|_4^4 \\ &\leq 4\|\Delta f\|^2 + \|\Delta^2 u\|^2 + 8\|\Delta u\|^2 + C\|\nabla u\|^4 + C\|\nabla u\|^8 \end{aligned}$$

where the constant  $C > 0$  is due to the boundedness of  $\Omega$  and the imbedding  $H_0^1 \hookrightarrow L^\infty$ . As in lemma 5.3, the following estimates can be justified for the Galerkin approximations of  $u$  on the Galerkin basis given by the eigenfunctions of the Laplacian with Dirichlet boundary conditions. Now we employ the imbedding  $H^4 \hookrightarrow H^3$  in the second term above, then we employ remark 5.6 to bound the sum  $8\|\Delta u\|^2 + C\|\nabla u\|^4 + C\|\nabla u\|^8$  independently of  $t > 0$ ; and since  $u_0 \in H^3 \cap H_0^1$ , for all  $t \geq 0$ , there is a generic constant  $C$  sufficiently large, but independent of  $t$ , such that

$$\frac{d}{dt} \|\nabla \Delta u\|^2 + c\|\nabla \Delta u\|^2 \leq C.$$

With the aid of [24, Prop. 2.64], we conclude that the ball  $B_3^0(0, \sqrt{C/c})$  in  $H^3 \cap H_0^1$  is positively invariant, and that, the ball  $B_3^0(0, \rho_3^p)$ , for any  $\rho_3^p > \sqrt{C/c}$ , is absorbing and positively invariant.  $\square$

*Lemma 5.12.* For any  $T > 0$ ,  $u_0 \in H^3 \cap H_0^1$ , and  $f \in H^2$ ,

$$u \in C([0, T]; H^3 \cap H_0^1).$$

*Proof.* Integrating (71) on  $[0, T]$  shows  $u \in L^2(0, T; H^4)$ . After (formally) multiplying (54) in  $L^2$  by  $2(-\Delta)u(t)$ , integrating on  $[0, T]$  shows  $u_t \in L^2(0, T; H^2)$ . The claim then follows by [21, §1.2, 1.3] (or see [24, Thm. A.80]).  $\square$

Lemmas 5.10 and 5.12 complete hypothesis 3.8.

We conclude that the parabolic semiflow  $S_p$  admits a compact inertial manifold  $\widehat{\mathcal{M}}_3^0$  in  $L^2$  satisfying (27). Let  $\mathcal{W}^p$  be a bounded subset of  $\mathfrak{I}_1$  and define  $C_0^0 := \mathcal{M}_{\mathcal{W}^p}^0$  as in (9). Then the sets

$$C_{j+1}^0 := \bigcup_{t \geq c_j^p} K_p(t)C_j^0$$

are bounded in  $X_{j+1}^0$ , for  $j = 0, 1, 2$ . Let

$$\omega_{c_2^p}^{K_p}(C_2^0) := \bigcup_{t \geq c_2^p} K_p(t)C_2^0$$

and  $u_0 \in \omega_{c_2^p}^{K_p}(C_2^0)$ . Set

$$\tau_3^p := \begin{cases} 0 & \text{if } \omega_{c_2^p}^{K_p}(C_2^0) \subseteq G_3^0 \\ \frac{1}{c} \ln \left( \frac{\|u_0\|_3^2 - 1}{(\rho_3^p)^2 - \frac{C}{c}} \right) & \text{otherwise.} \end{cases}$$

The constants  $C, c > 0$  were given in the proof of lemma 5.11 and  $\rho_3^p > 0$  was given in lemma 5.11. Then we are ready to conclude that the set defined by

$$\widehat{\mathcal{M}}_3^0 = \bigcup_{\tau \in I_3^p} S_p(\tau) \omega_{c_2^p}^{K_p}(C_2^0),$$

is a compact inertial manifold for  $S_p$  in  $L^2$ .

**5.3. The hyperbolic problem.** In the hyperbolic case, problem (45)-(47) is written as the abstract Cauchy problem:

$$(72) \quad \frac{dU(t)}{dt} + A_\varepsilon U(t) = F_\varepsilon(U(t)), \quad t > 0$$

with the initial condition

$$U(0) = (u_0, u_1)^T,$$

where

$$U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad A_\varepsilon := \begin{pmatrix} 0 & -\frac{1}{\varepsilon} \\ -\Delta & \frac{1}{\varepsilon} \end{pmatrix}, \quad F_\varepsilon(U) := \begin{pmatrix} 0 \\ f - u^3 + u \end{pmatrix},$$

and  $u_0 \in H_0^1$ ,  $u_1, f \in L^2$ . After we establish hypotheses 2.1, 2.2, 3.2, 3.5, 3.6, 3.8, and 3.10, which are enough to show the existence of a family of compact inertial manifolds  $(\widehat{\mathcal{M}}_3^\varepsilon)_{\varepsilon \in (0,1]}$  in  $X_1^\varepsilon$  for the hyperbolic case, we will return to hypothesis 4.1 and a weaker version of hypothesis 4.6. Note that hypothesis 4.5 is not used in the hyperbolic case.

In a similar fashion, the spaces defined above by  $X_k = H_k \times H_{k-1}$ ,  $k = 1, 2, 3$ , satisfy hypothesis 3.5. The operator  $-A_\varepsilon$  above generates a  $C^0$ -semigroup on  $X_1^\varepsilon$  (cf. [4, §7.4, Thm. 6], or [23, §12.2, Thm. 3] with [17, Thm. 6.1, p. 38]). It is well-known that the eigenfunctions of the Laplacian form an orthogonal basis in  $H_0^1$  which is orthonormal in  $L^2$  (cf. e.g. [4, §6.5, Thm. 1]). (Thus, the product space  $X_1^\varepsilon$  contains a countable orthogonal basis, and  $X_1^\varepsilon$  is separable.) We also know that the modified nonlinearity

$$(73) \quad \tilde{F}_\varepsilon(U) = (0, f - (\gamma(u))^3 + \gamma(u))^T,$$

where  $\gamma$  was given above, is both globally bounded and Lipschitz continuous on  $X_1^\varepsilon$ . So far this establishes hypothesis 2.1. With this, we then show that for  $\varepsilon$  sufficiently small, the operator  $A_\varepsilon$  satisfies the spectral gap condition relative to the modified nonlinearity  $\tilde{F}_\varepsilon(U)$ .

*Lemma 5.13.* The map  $\tilde{F}_\varepsilon(U) : X_1^\varepsilon \rightarrow X_1^\varepsilon$  defined in (73) is globally bounded and Lipschitz continuous with Lipschitz constant  $\ell_{\tilde{F}_0} := 1 + 3(2\delta - 1)^2$  (this is the same constant used for the parabolic problem given in (50)).

*Proof.* Since

$$\|\tilde{F}_\varepsilon(U)\|_{X_1^\varepsilon}^2 = \|\tilde{F}_0(u)\|^2,$$

the claims follow from lemma 5.1.  $\square$

Our aim now is to show that when  $\varepsilon$  is sufficiently small,  $A_\varepsilon$  satisfies the spectral gap condition relative to the modified nonlinearity  $\tilde{F}_\varepsilon$ . Following [24, §5.7.2], introduce the “time rescaled” unknown

$$w(t, x) := u(\sqrt{\varepsilon}t, x);$$

$w$  solves the equation

$$w_{tt} + \frac{1}{\sqrt{\varepsilon}}w_t - \Delta w = f - g(w).$$

We set  $\alpha := \frac{1}{2\sqrt{\varepsilon}}$  and consider the equation (after re-naming  $w$  back to  $u$ )

$$u_{tt} + 2\alpha u_t - \Delta u = f - g(u).$$

This equation is equivalent to the Cauchy problem (72) where now

$$A_\varepsilon = \begin{pmatrix} 0 & -1 \\ -\Delta & 2\alpha \end{pmatrix}.$$

Observe that the eigenvalue equation,

$$A_\varepsilon U = \mu U, \quad U = (u, v) \in X_1^\varepsilon,$$

is equivalent to the system

$$\begin{cases} v &= -\mu u \\ -\Delta u + 2\alpha v &= \mu v. \end{cases}$$

Thus  $u \in H_0^1$  satisfies  $-\Delta u = (2\alpha\mu - \mu^2)u$  and  $2\alpha\mu - \mu^2$  is an eigenvalue of  $-\Delta$  with respect to Dirichlet boundary conditions. As usual, let us denote by  $(\lambda_j)_{j \in \mathbb{N}}$  the sequence these eigenvalues, then

$$2\alpha\mu - \mu^2 = \lambda_j, \quad j \in \mathbb{N}.$$

For each  $j \in \mathbb{N}$  this equation has two solutions given by

$$\mu = \alpha \pm \sqrt{\alpha^2 - \lambda_j}.$$

Recall that the eigenvalues of  $-\Delta$  with respect to Dirichlet boundary conditions on  $\Omega = (0, \pi)$  satisfy

$$0 < \lambda_1 = 1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots .$$

So when  $\alpha^2 < 1$ , then  $\alpha^2 < \lambda_j$  for all  $j \in \mathbb{N}$  and all the eigenvalues of  $A_\varepsilon$  are complex with the same real part; precisely,  $\Re(\mu) = \alpha$ , and consequently, the spectral gap condition cannot hold. On the other hand, when  $\alpha^2 \geq \lambda_j$  for some  $j \in \mathbb{N}$ , i.e.  $\varepsilon \leq \frac{1}{4\lambda_j} \leq \frac{1}{4}$ , the first  $j$  eigenvalues of  $A_\varepsilon$  are real, with at least two being distinct, and we may investigate whether condition (5) holds. We cite [24, Thm. 5.44] where it is shown that there exists  $\varepsilon_s \in (0, \frac{1}{4}]$  such that for all  $\varepsilon \in (0, \varepsilon_s]$ ,  $A_\varepsilon$  satisfies condition (5) relative to  $\tilde{F}_\varepsilon$ .

This establishes hypothesis 2.2 (without parts 3 & 4); indeed,  $A_\varepsilon$  is not self-adjoint (cf. e.g. [1, Lem. 3.1]) so part 4 of hypothesis 2.2 cannot hold. Define the Lipschitz continuous hyperbolic semiflow  $S_\varepsilon(t) : X_1^\varepsilon \rightarrow X_1^\varepsilon$  through the mild solution by

$$(74) \quad S_\varepsilon(t)(u_0, u_1)(x) := (u(t, x, u_0, u_1), u_t(t, x, u_0, u_1)).$$

By [24, Thm. 5.44] there is an integer  $N_\varepsilon \geq 1$  so that condition (5) holds. With this set  $\Sigma_1 := \text{span}\{\omega_n(x)\}_{n=1}^{N_\varepsilon}$ , where  $\omega_n(x)$ , for  $n = 1, 2, 3, \dots$ , are the eigenvectors of  $A_\varepsilon$  in  $X_1$  with domain  $X_2$ . Also set  $\Sigma_2 := \Sigma_1^\perp$ . Then there exists a map

$$m_\varepsilon \in \mathcal{G}_{\ell_{\tilde{F}_0}} := \{\Psi \in C_b(\Sigma_1; \Sigma_2) : \|\Psi(\chi) - \Psi(\psi)\|_{X_1^\varepsilon} \leq \ell_F \|\chi - \psi\|_{X_1^\varepsilon}, \forall \chi, \psi \in \Sigma_1\}$$

such that the subset of  $X_1^\varepsilon$  defined by the graph of  $m_\varepsilon$  on  $\Sigma_1$ ,

$$(75) \quad \mathcal{M}_1^\varepsilon := \text{graph}(m_\varepsilon) = \{\xi + m_\varepsilon(\xi) : \xi \in \Sigma_1\},$$

is an inertial manifold for  $S_p$  in  $X_1^\varepsilon$ . Again, a specific choice for  $\delta > 1$  will be made later after we show the existence of the absorbing set  $G_3^\varepsilon$  per hypothesis 3.10.

To prepare for the compatibility condition later, denote by  $N_{\varepsilon*}$  the minimum integer  $N_\varepsilon$  such that condition (5) holds and define

$$N^* := \max\{N_{p*}, N_{\varepsilon*}\}.$$

For the hyperbolic case, we know that for any  $T > 0$ ,  $u_0 \in H_0^1$ ,  $u_1 \in L^2$ , and  $f \in L^2$ , the weak solution the hyperbolic problem

$$(76) \quad \begin{cases} \varepsilon u_{tt} + u_t - \Delta u + u^3 - u = f \\ u(t, 0) = u(t, \pi) = 0 \\ u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1 \end{cases}$$

satisfies

$$(77) \quad u \in C([0, T]; H_0^1) \cap C^1([0, T]; L^2),$$

and when  $u_0 \in H^2 \cap H_0^1$ ,  $u_1 \in H_0^1$ , and  $f \in H^1$ , the weak solution to the hyperbolic problem satisfies

$$(78) \quad u \in C([0, T]; H^2 \cap H_0^1) \cap C^1([0, T]; H_0^1) \cap C^2([0, T]; L^2).$$

Moreover, the function  $t \mapsto \|\nabla u(t)\| + \|u_t(t)\| + \|\varepsilon u_{tt}(t)\|$  is absolutely continuous on  $[0, T]$  (cf. [24, Thm. 3.20]). From (77) and (78), we infer that the semiflow  $S_\varepsilon$  defined by (74) is continuous in  $X_1^\varepsilon$  and  $X_2^\varepsilon$ , leaving us to later verify hypothesis 3.8 for  $j = 2$ .

In the hyperbolic setting, we know the continuous semiflow  $S_\varepsilon$  admits a decomposition in  $X_1^\varepsilon$  in accordance with hypothesis 3.2. Define the one-parameter family of maps,  $Z_\varepsilon(t) : X_1^\varepsilon \rightarrow X_1^\varepsilon$ , by

$$Z_\varepsilon(t)(u_0, u_1)(x) := (v(t, x, u_0, u_1), v_t(t, x, u_0, u_1)),$$

where  $v(\cdot, \cdot, u_0, u_1)$  is a solution of the IBVP

$$(79) \quad \begin{cases} \varepsilon v_{tt} + v_t - \Delta v + v^3 = 0 & \text{in } Q \\ v|_{\partial\Omega} = 0, \\ v(0, \cdot) = u_0, \quad v_t(0, \cdot) = u_1. \end{cases}$$

With such  $v$ , we may define a second function  $w(\cdot, \cdot, u_0, u_1)$  as the solution of the IBVP (recall  $g(u) = u^3 - u$ )

$$(80) \quad \begin{cases} \varepsilon w_{tt} + w_t - \Delta w = f + v^3 - g(v + w) & \text{in } Q \\ w|_{\partial\Omega} = 0, \\ w(0, \cdot) = 0, \quad w_t(0, \cdot) = 0. \end{cases}$$

Through the dependence of  $w$  on  $v$  and  $(u_0, u_1)$ , problem (80) then defines a one-parameter family of maps,  $K_\varepsilon(t) : X_1^\varepsilon \rightarrow X_1^\varepsilon$ , defined by

$$K_\varepsilon(t)(u_0, u_1)(x) := (w(t, x, u_0, u_1), w_t(t, x, u_0, u_1)).$$

Notice that if  $v$  and  $w$  are solutions to problems (79) and (80) respectively, then  $u = v + w$  is a solution to (76). We will show that the family of maps  $K_\varepsilon$  are uniformly compact for each  $\varepsilon \in (0, 1]$  below. For the remaining portions of hypothesis 3.2, we refer to [24, §3.4.3] or [33, Thm. 6.4.4]; i.e., we know that the solution  $(v(t), v_t(t))$  is uniformly bounded (and decaying) in  $X_1^\varepsilon$  if  $(u_0, u_1) \in X_1^\varepsilon$  and  $f \in L^2$ . Additionally, we know that for such data, the solution  $(u(t), u_t(t))$  is also uniformly bounded in  $X_1^\varepsilon$  ([24, see (3.74) in the proof of Thm. 3.21]).

*Lemma 5.14.* Let  $(u_0, u_1) \in X_1^\varepsilon$ , and  $f \in L^2$ . Then the map  $t \mapsto (w(t), w_t(t))$  is uniformly bounded (with respect to  $t$  and  $\varepsilon$ ) in  $X_2^\varepsilon$ .

*Proof.* In a similar fashion to [24, Prop. 4.9], formally multiply the PDE in (80) by  $-2\Delta w_t(t) - \Delta w(t)$  in  $L^2$  and add  $2\alpha\langle f, \Delta w(t) \rangle$ , where  $\alpha > 0$  is yet to be determined, to both sides to obtain (with  $v + w = u$ )

$$(81) \quad \begin{aligned} \frac{d}{dt}(N_2 + 2\langle f, \Delta w \rangle) + (2 - \varepsilon)\|\nabla w_t\|^2 + \|\Delta w\|^2 + 2\alpha\langle f, \Delta w \rangle \\ = \langle(2\alpha - 1)f, \Delta w \rangle + \langle v^3 - u^3 + u, -2\Delta w_t - \Delta w \rangle, \end{aligned}$$

where

$$N_2(w(t), w_t(t)) := \varepsilon\|\nabla w_t(t)\|^2 + \varepsilon\langle\nabla w_t(t), \nabla w(t)\rangle + \frac{1}{2}\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2$$

is, as we shall see, the square of an equivalent norm in  $X_2^\varepsilon$ . To control the right hand side of (81), first

$$\langle(2\alpha - 1)f, \Delta w \rangle \leq (2\alpha - 1)^2\|f\|^2 + \frac{1}{4}\|\Delta w\|^2.$$

Next

$$\langle v^3 - u^3 + u, -2\Delta w_t \rangle = 2\langle 3v^2\nabla v - 3u^2\nabla u + \nabla u, \nabla w_t \rangle \leq C_1 + \frac{1}{2}\|\nabla w_t\|^2,$$

where the constant  $C_1 > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$  and the uniform bound of  $v(t)$  and  $u(t)$  in  $H_0^1$ . Last,

$$\langle v^3 - u^3 + u, -\Delta w \rangle \leq 3|v|_6^6 + 3|u|_6^6 + 3\|\nabla u\|^2 + \frac{1}{4}\|\Delta w\|^2 \leq C_2 + \frac{1}{4}\|\Delta w\|^2,$$

where  $C_2 > 0$  depends on the measure of  $\Omega$  (i.e.  $\pi$ ), the imbedding  $H_0^1 \hookrightarrow L^\infty$ , and the uniform bound on  $v(t)$  and  $u(t)$  in  $H_0^1$ . We also employed the Poincaré inequality  $\|u\| \leq \|\nabla u\|$ ; i.e. with  $\Omega = (0, \pi)$ , the Poincaré constant is 1 (cf. [20, Exercise 7.3] and [2, Ex. 2.19]). Now we rewrite (81) as

$$(82) \quad \frac{d}{dt}(N_2 + 2\langle f, \Delta w \rangle) + \frac{1}{2}(\varepsilon\|\nabla w_t\|^2 + \|\Delta w\|^2) + 2\alpha\langle f, \Delta w \rangle \leq C_3(\alpha).$$

Since

$$\frac{1}{5}N_2 \leq \frac{1}{2}(\varepsilon\|\nabla w_t\|^2 + \|\Delta w\|^2),$$

then (82) becomes

$$\frac{d}{dt}(N_2 + 2\langle f, \Delta w \rangle) + \frac{1}{5}(N_2 + 2\langle f, \Delta w \rangle) \leq C_3,$$

when we choose  $\alpha = \frac{1}{5}$ . The constant  $C_3 > 0$  above is independent of  $t$  and  $\varepsilon$ . Since  $\frac{1}{2}\|(w(t), w_t(t))\|_{X_2^\varepsilon}^2 \leq N_2$ , we deduce

$$\|(w(t), w_t(t))\|_{X_2^\varepsilon}^2 \leq 2e^{-t/5}(N_2(0) + 2\langle f, \Delta w_0 \rangle) + 10C_3(1 - e^{-t/5}).$$

Since  $(w_0, w_1) = (0, 0)$ , this simplifies to,

$$\|(w(t), w_t(t))\|_{X_2^\varepsilon}^2 \leq 10C_3 = C,$$

from which the claim follows.  $\square$

Let  $B$  be a bounded subset of  $X_1^\varepsilon$ . It follows from lemma 5.14 that the set

$$\bigcup_{t \geq 0} K_\varepsilon(t)B$$

is bounded, independent of  $t$  and  $\varepsilon$ , in  $X_2^\varepsilon$ ; hence, the family of maps  $K_\varepsilon$  is uniformly compact in  $X_1^\varepsilon$ . This establishes hypothesis 3.2. Moreover, we have shown

hypothesis 3.6 holds for  $k = 1$  when  $c_1^\varepsilon = 0$ . It remains to establish hypotheses 3.6 for  $k = 2$ , and 3.8 for  $j = 1$ , as well as hypothesis 3.10.

*Lemma 5.15.* Let  $(u_0, u_1) \in X_2^\varepsilon$  and  $f \in H^1$ . Then the map  $t \mapsto (v(t), v_t(t))$  is uniformly bounded in  $X_2^\varepsilon$ .

*Proof.* Multiplying the equation (79) by  $-2\Delta v_t(t) - \Delta v(t)$  in  $L^2$  yields

$$(83) \quad \frac{d}{dt} N_2 + (2 - \varepsilon) \|\nabla v_t\|^2 + \|\Delta v\|^2 = \langle -v^3, -2\Delta v_t - \Delta v \rangle.$$

Here,  $N_2 = N_2(v(t), v_t(t))$ . The right hand side is controlled by, first

$$\langle -v^3, -2\Delta v_t \rangle = -6 \langle v^2 \nabla v, \nabla v_t \rangle \leq 18|v|_\infty^2 \|\nabla v\|^2 + \frac{1}{2} \|\nabla v_t\|^2 \leq C_1 + \frac{1}{2} \|\nabla v_t\|^2,$$

where the constant  $C_1 > 0$  depends on the imbedding of  $H_0^1 \hookrightarrow L^\infty$  and the uniform bound of  $v(t)$  in  $H_0^1$ . Second,

$$-\langle v^3, -\Delta v \rangle \leq \frac{1}{2} |v|_6^6 + \frac{1}{2} \|\Delta v\|^2 \leq C_2 + \frac{1}{2} \|\Delta v\|^2$$

where  $C_2 > 0$  depends on  $m(\Omega)$ , the imbedding of  $H_0^1 \hookrightarrow L^\infty$ , and the uniform bound of  $v(t)$  in  $H_0^1$ . With this, (83) becomes

$$\frac{d}{dt} N_2 + \frac{1}{2} (\varepsilon \|\nabla v_t\|^2 + \|\Delta v\|^2) \leq C_3.$$

The constant  $C_3 > 0$  is independently of  $t$  and  $\varepsilon$ . We know that

$$\frac{1}{5} N_2 \leq \frac{1}{2} (\varepsilon \|v_t\|^2 + \|\Delta v\|^2),$$

so

$$\frac{d}{dt} N_2 + \frac{1}{5} N_2 \leq C_3.$$

The fact that  $\frac{1}{2} \|(v(t), v_t(t))\|_{X_2^\varepsilon}^2 \leq N_2$  yields

$$\|(v(t), v_t(t))\|_{X_2^\varepsilon}^2 \leq 2e^{-t/5} N_2(0) + 10C_3(1 - e^{-t/5}).$$

Thus,

$$\|(v(t), v_t(t))\|_{X_2^\varepsilon}^2 \leq \begin{cases} 10C_3 & \text{if } N_2(0) \leq 5C_3 \\ 2N_2(0) & \text{otherwise.} \end{cases}$$

This proves the claim.  $\square$

*Corollary 5.16.* For any  $(u_0, u_1) \in X_2^\varepsilon$  and  $f \in H^1$ , the solution  $(u(t), u_t(t))$  of the hyperbolic problem (76) is uniformly bounded (with respect to  $t$  and  $\varepsilon$ ) in  $X_2^\varepsilon$ .

*Proof.* Since  $u = v + w$ , the result follows by lemmas 5.14 and 5.15.  $\square$

*Lemma 5.17.* Let  $(u_0, u_1) \in X_2^\varepsilon$ , and  $f \in H^1$ . Then the map  $t \mapsto (w(t), w_t(t))$  is uniformly bounded (with respect to  $t$  and  $\varepsilon$ ) in  $X_3^\varepsilon$ .

*Proof.* Formally multiply the PDE in (80) by  $2(-\Delta)^2 w_t(t) + (-\Delta)^2 w(t)$  in  $L^2$  and add  $2\alpha \langle \nabla f, \nabla \Delta w(t) \rangle$ , where  $\alpha > 0$  is to be determined, to both sides to obtain

$$(84) \quad \begin{aligned} \frac{d}{dt} (N_2 + 2\alpha \langle \nabla f, \nabla \Delta w \rangle) + (2 - \varepsilon) \|\Delta w_t\|^2 + \|\nabla \Delta w\|^2 + 2\alpha \langle \nabla f, \nabla \Delta w \rangle \\ = \langle (2\alpha - 1) \nabla f, \nabla \Delta w \rangle + \langle v^3 - u^3 + u, 2(-\Delta)^2 w_t + (-\Delta)^2 w \rangle \end{aligned}$$

where

$$(85) \quad N_3(w(t), w_t(t)) := \varepsilon \|\Delta w_t(t)\|^2 + \varepsilon \langle \Delta w_t(t), \Delta w(t) \rangle + \frac{1}{2} \|\Delta w(t)\|^2 + \|\nabla \Delta w(t)\|^2$$

is the square of an equivalent norm in  $X_3^\varepsilon$ . To control the right hand side of (84), first

$$\langle (2\alpha - 1)\nabla f, \nabla \Delta w \rangle \leq (2\alpha - 1)^2 \|\nabla f\|^2 + \frac{1}{4} \|\nabla \Delta w\|^2.$$

Next

$$\begin{aligned} \langle v^3 - u^3 + u, 2(-\Delta)^2 w_t \rangle &= 2\langle 6v(\nabla v)^2 + 3v^2 \Delta v - 6u(\nabla u)^2 - 3u^2 \Delta u + \Delta u, \Delta w_t \rangle \\ &\leq C_1 + \frac{1}{2} \|\Delta w_t\|^2, \end{aligned}$$

where the constant  $C_1 > 0$  depends on the imbedding  $H^2 \cap H_0^1 \hookrightarrow L^\infty$  and the uniform bound of  $v(t)$  and  $u(t)$  in  $H^2 \cap H_0^1$  (see lemma 5.15 and corollary 5.16 respectively). Last,

$$\langle v^3 - u^3 + u, (-\Delta)^2 w \rangle = \langle 3v^2 \nabla v - 3u^2 \nabla u + \nabla u, \nabla \Delta w \rangle \leq C_2 + \frac{1}{4} \|\nabla \Delta w\|^2,$$

where  $C_2 > 0$  depends on the imbedding  $H^2 \cap H_0^1 \hookrightarrow L^\infty$  and the uniform bound on  $v(t)$  and  $u(t)$  in  $H^2 \cap H_0^1$ . Now we rewrite (81) as

$$(86) \quad \frac{d}{dt} (N_3 + 2\langle \nabla f, \nabla \Delta w \rangle) + \frac{1}{2} (\varepsilon \|\Delta w_t\|^2 + \|\nabla \Delta w\|^2) + 2\alpha \langle \nabla f, \nabla \Delta w \rangle \leq C_3(\alpha).$$

Since

$$\frac{1}{5} N_3 \leq \frac{1}{2} (\|\Delta w_t\|^2 + \|\nabla \Delta w\|^2),$$

then (86) becomes

$$\frac{d}{dt} (N_3 + 2\langle \nabla f, \nabla \Delta w \rangle) + \frac{1}{5} (N_3 + 2\langle \nabla f, \nabla \Delta w \rangle) \leq C_3,$$

when we choose  $\alpha = \frac{1}{5}$  again. The constant  $C_3 > 0$  above is independent of  $t$  and  $\varepsilon$ . Since  $\frac{1}{2} \|(w(t), w_t(t))\|_{X_3^\varepsilon}^2 \leq N_3$ , it follows that

$$\|(w(t), w_t(t))\|_{X_3^\varepsilon}^2 \leq 2e^{-t/5} (N_3(0) + 2\langle \nabla f, \nabla \Delta w_0 \rangle) + 10C_3(1 - e^{-t/5});$$

again  $(w_0, w_1) = (0, 0)$ , so that

$$\|(w(t), w_t(t))\|_{X_3^\varepsilon}^2 \leq 10C_3 = C.$$

Therefore the claim holds.  $\square$

Consequently, we can choose  $c_2^\varepsilon = 0$  which verifies hypothesis 3.6 for the hyperbolic problem. It now remains to show hypothesis 3.8 for  $j = 1$  as well as hypothesis 3.10. Both claims will follow naturally from the following one.

*Lemma 5.18.* For any  $(u_0, u_1) \in X_3^\varepsilon$  and  $f \in H^2$ , the solution  $(u(t), u_t(t))$  of the hyperbolic problem (76) is uniformly bounded (with respect to  $t$  and  $\varepsilon$ ) in  $X_3^\varepsilon$ .

*Proof.* Formally multiply the equation (76) by  $2(-\Delta)^2 u_t(t) + (-\Delta)^2 u(t)$  in  $L^2$  to obtain

$$(87) \quad \frac{d}{dt} N_3 + (2 - \varepsilon) \|\Delta u_t\|^2 + \|\nabla \Delta u\|^2 = \langle f - u^3 + u, 2(-\Delta)^2 u_t + (-\Delta)^2 u \rangle,$$

where

$$N_3(u(t), u_t(t)) := \varepsilon \|\Delta u_t(t)\|^2 + \varepsilon \langle \Delta u_t(t), \Delta u(t) \rangle + \frac{1}{2} \|\Delta u(t)\|^2 + \|\nabla \Delta u(t)\|^2$$

satisfies

$$\frac{1}{2} \|(u(t), u_t(t))\|_{X_3^\varepsilon}^2 \leq N_3(u(t), u_t(t)) \leq \frac{5}{2} \|(u(t), u_t(t))\|_{X_3^\varepsilon}^2.$$

Returning to the right hand side of (87), first

$$(88) \quad \begin{aligned} \langle f - u^3 + u, 2(-\Delta)^2 u_t \rangle &= 2\langle \Delta f - 6u(\nabla u)^2 - 3u^2 \Delta u + \Delta u, \Delta u_t \rangle \\ &\leq 4\|\Delta f\|^2 + C_1 + \frac{1}{2}\|\Delta u_t\|^2, \end{aligned}$$

where the constant  $C_1 > 0$  depends on the imbedding  $H^2 \cap H_0^1 \hookrightarrow L^\infty$  and the uniform bound of  $u(t)$  in  $H^2 \cap H_0^1$  (cf. corollary 5.16). Next

$$(89) \quad \begin{aligned} \langle f - u^3 + u, (-\Delta)^2 u \rangle &= \langle \nabla f - 3u^2 \nabla u + \nabla u, \nabla \Delta u \rangle \\ &\leq \frac{5}{2}\|\nabla f\|^2 + C_2 + \frac{1}{2}\|\nabla \Delta u\|^2, \end{aligned}$$

where  $C_2 > 0$  depends on similar parameters as  $C_1$ . With these, (87) becomes

$$\frac{d}{dt}N_3 + \frac{1}{5}N_3 \leq C_3,$$

from which we deduce

$$(90) \quad \begin{aligned} \|(u(t), u_t(t))\|_{X_3^\varepsilon}^2 &\leq 2e^{-t/5}(N_2(0) - 5C_3) + 10C_3 \\ &\leq \begin{cases} 10C_3 & \text{if } N_2(0) \leq 5C_3 \\ 2N_2(0) & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore the claim holds.  $\square$

*Lemma 5.19.* For any  $T > 0$ ,  $(u_0, u_1) \in X_3^\varepsilon$ , and  $f \in H^2$ ,

$$u \in C([0, T]; H^3 \cap H_0^1) \cap C^1([0, T]; H^2 \cap H_0^1) \cap C^2([0, T]; H_0^1).$$

*Proof.* Formally multiply the PDE in (76) by  $2(-\Delta)^2 u_t(t) + (-\Delta)^2 u(t)$  in  $L^2$  to obtain (87), (88), and (89), we have

$$\frac{d}{dt}N_3 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla \Delta u\|^2 \leq \frac{13}{2}\|\nabla f\|^2 + C_1 + C_2.$$

Integrating with respect to  $t$  on  $[0, T]$  and omitting the terms  $N_3(t)$  and  $\frac{1}{2}\|\Delta u_t\|$ , we see  $u_t \in L^2(0, T; H^3)$ . By omitting  $\frac{1}{2}\|\nabla \Delta u\|^2$  instead of  $\frac{1}{2}\|\Delta u_t\|^2$  we obtain  $u_t \in L^2(0, T; H^2)$ , so by the trace theorem (cf. [21])  $u \in C([0, T]; H^2)$ . To obtain  $u \in C([0, T]; H^3)$ , follow the arguments made in e.g. [28].  $\square$

This completes the justification of hypothesis 3.8. We proceed to show hypothesis 3.10.

*Lemma 5.20.* For all  $(u_0, u_1) \in X_3^\varepsilon$  and  $f \in H^2$ , there exists  $\rho_3^\varepsilon > 0$  such that the ball

$$G_3^\varepsilon = B_3^\varepsilon(0, \rho_3^\varepsilon) := \{z \in X_3^\varepsilon : \|z\|_{X_3^\varepsilon} \leq \rho_3^\varepsilon\}$$

is absorbing and positively invariant for  $S_\varepsilon$ .

*Proof.* Let  $\rho_3^\varepsilon > 0$  be such that  $(\rho_3^\varepsilon)^2 > 10C_3$ . Using (90): when  $N_3(0) > 5C_3$  and  $2N_3(0) \leq (\rho_3^\varepsilon)^2$ , then  $(u(t), u_t(t)) \in B_3^\varepsilon(0, \rho_3^\varepsilon)$  for all  $t \geq \tau_3^\varepsilon = 0$ . Otherwise, when  $N_3(0) > 5C_3$  and  $2N_3(0) > (\rho_3^\varepsilon)^2$ , then  $(u(t), u_t(t)) \in B_3^\varepsilon(0, \rho_3^\varepsilon)$  for all  $t \geq \tau_3^\varepsilon$  where

$$\tau_3^\varepsilon := 5 \ln \left( \frac{2(N_3(0) - 5C_3)}{(\rho_3^\varepsilon)^2 - 10C_3} \right).$$

On the other hand, if  $N_3(0) \leq 5C_3$ , then  $2N_3(0) \leq 10C_3 < (\rho_3^\varepsilon)^2$  and, trivially, for all  $t \geq 0$ ,  $\|(u(t), u_t(t))\|_{X_3^\varepsilon} \leq \rho_3^\varepsilon$ .  $\square$

We conclude that the hyperbolic semiflow  $S_\varepsilon$  admits a compact inertial manifold  $\widehat{\mathcal{M}}_3^\varepsilon$  in  $X_1^\varepsilon$  satisfying (28). Let  $\mathcal{W}^\varepsilon$  be a bounded subset of  $\Sigma_1$  and define  $C_1^\varepsilon := \mathcal{M}_{\mathcal{W}^\varepsilon}^\varepsilon$  as in (9). Then the sets

$$C_2^\varepsilon := \bigcup_{t \geq c_1^\varepsilon} K_\varepsilon(t)C_1^\varepsilon \text{ and } C_3^\varepsilon := \bigcup_{t \geq c_2^\varepsilon} K_\varepsilon(t)C_2^\varepsilon$$

are bounded in, respectively,  $X_2^\varepsilon$  and  $X_3^\varepsilon$ . Let

$$\omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon) := \bigcup_{t \geq c_2^\varepsilon} K_\varepsilon(t)C_2^\varepsilon$$

and  $(u_0, u_1) \in \omega_{c_1^\varepsilon}^{K_\varepsilon}(C_1^\varepsilon)$ . Set

$$\tau_3^\varepsilon := \begin{cases} 0 & \text{if } \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2) \subseteq G_3^\varepsilon \\ 5 \ln \left( \frac{2(N_3(0) - 5C_3)}{(\rho_3^\varepsilon)^2 - 10C_3} \right) & \text{otherwise.} \end{cases}$$

The constant  $C_3$  was given in the proof of lemma 5.20 and  $\rho_3^\varepsilon$  was given in lemma 5.20. Recall the equivalent norm  $N_3$  was defined in (85). Then we are ready to conclude that, for each  $\varepsilon \in (0, \varepsilon_s]$  the set defined by,

$$\widehat{\mathcal{M}}_3^\varepsilon = \bigcup_{\tau \in I_3^\varepsilon} S_\varepsilon(\tau) \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon),$$

is a compact inertial manifold for  $S_\varepsilon$  in  $X_1^\varepsilon$ .

**5.4. The weak robustness result.** We have established hypotheses 2.1, 2.2, 3.2, 3.5, 3.6, 3.8, 3.10, and 4.5 for both the parabolic problem and its perturbed hyperbolic relaxation. Finally, we move onto hypothesis 4.1.

*Lemma 5.21.* The map  $\mathcal{E} : H^2 \cap H_0^1 \rightarrow L^2$  defined by

$$u \mapsto f + \Delta u - u^3 + u$$

is locally Lipschitz continuous (recall  $Y_2 = H^2 \cap H_0^1$ ,  $Y_1 = H_0^1$ , and  $Y_0 = L^2$ ).

*Proof.* First, recall that the map  $g(u) := u^3 - u$  is locally Lipschitz continuous from  $H_0^1$  into  $L^2$ . Let  $M > 0$  and  $u, v \in H^2 \cap H_0^1$  be such that  $\|u\|_2 \leq M$  and  $\|v\|_2 \leq M$ . Then

$$\|\mathcal{E}(u) - \mathcal{E}(v)\| \leq \|\Delta u - \Delta v\| + \|g(u) - g(v)\| \leq C\|u - v\|_2,$$

where  $C > 0$  depends on the  $M$ , the local Lipschitz constant of  $g$ , and the imbedding  $H^2 \hookrightarrow H_0^1$ .  $\square$

Let  $G_3^\varepsilon$  be defined as in lemma 5.20 and  $(h_0, h_1) \in G_3^\varepsilon$  (and recall that on  $G_3^\varepsilon$ , the modified nonlinearity  $\widetilde{F}_\varepsilon$  is the same as the original  $F_\varepsilon$ ). Solutions  $h(t)$  of “problem  $(H_\varepsilon)$ ,”

$$\begin{cases} \varepsilon h_{tt} + h_t - \Delta h + h^3 - h = f & \text{in } Q \\ h|_{\partial\Omega} = 0 \\ h(0, \cdot) = h_0, h_t(0, \cdot) = h_1, \end{cases}$$

define the semiflow for the hyperbolic case ( $\varepsilon \in (0, \varepsilon_1]$ ),

$$S_\varepsilon(t)(h_0, h_1)(x) := (h(t, x, h_0, h_1), h_t(t, x, h_0, h_1)).$$

For the parabolic case ( $\varepsilon = 0$ ), solutions  $p(t)$  of “problem (P),”

$$\begin{cases} p_t - \Delta p + p^3 - p = f & \text{in } Q \\ p|_{\partial\Omega} = 0 \\ p(0, \cdot) = p_0, \end{cases}$$

define the semiflow

$$S_p(t)p_0(x) := p(t, x, p_0).$$

The lift of  $S_p$  is given by

$$\mathcal{L}S_p(t)p_0(x) := (p(t, x, p_0), \mathcal{E}(p(t, x, p_0))).$$

We are interested in the difference  $z(t) := h(t) - p(t)$  when  $h_0 = p_0$ , which solves

$$(91) \quad \begin{cases} \varepsilon z_{tt} + z_t - \Delta z + g(h) - g(p) = -\varepsilon p_{tt} & \text{in } Q \\ z|_{\partial\Omega} = 0 \\ z(0, \cdot) = 0, \quad z_t(0, \cdot) = h_1 - \mathcal{E}(h_0). \end{cases}$$

To establish (26) of hypothesis 4.1, we begin with the following result. The constants appearing in the estimates below are independent of  $\varepsilon$ . In many cases they may arise from the following inequalities: imbeddings, Poincaré, Lipschitz continuity, or uniform bounds on solutions.

*Lemma 5.22.* For any  $p_0, f \in L^2$ , there holds for all  $t \geq 0$ ,

$$(92) \quad \|p(t)\|^2 \leq e^{-2t}\|p_0\|^2 + \frac{1}{2} \left( \|f\|^2 + \frac{9}{8}\pi \right) (1 - e^{-2t}).$$

*Proof.* Multiplying the PDE in problem (P) by  $2p(t)$  yields for almost all  $t \geq 0$ ,

$$(93) \quad \begin{aligned} \frac{d}{dt}\|p\|^2 + 2\|\nabla p\|^2 + 2|p|_4^4 &= 2\langle f + p, p \rangle \leq \|f\|^2 + 3\|p\|^2 \\ &\leq \|f\|^2 + \frac{9}{8}\pi + 2|p|_4^4; \end{aligned}$$

i.e.,

$$\frac{d}{dt}\|p\|^2 + 2\|p\|^2 \leq \|f\|^2 + \frac{9}{8}\pi,$$

from which (92) follows.  $\square$

*Corollary 5.23.* It follows from (92) that for any  $p_0, f \in L^2$ , there holds for all  $t \geq 0$ ,

$$(94) \quad \|p(t)\|^2 \leq \|p_0\|^2 + \frac{1}{2} \left( \|f\|^2 + \frac{9}{8}\pi \right).$$

*Lemma 5.24.* Let  $p_0 \in H_0^1$  and  $f \in L^2$ . For all  $t \geq 0$  there holds,

$$(95) \quad \int_0^t \|p_t(\tau)\|^2 d\tau \leq 3\|f\|^2 + (3 + C)\|\nabla p_0\|^2,$$

where  $C > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ .

*Proof.* Multiplying the PDE in problem (P) by  $2p_t(t)$  yields for almost all  $t \geq 0$ ,

$$(96) \quad \|p_t\|^2 + \frac{d}{dt} \left( \|\nabla p\|^2 + \frac{1}{2}|p|_4^4 - \|p\|^2 - 2\langle f, p \rangle \right) = 0.$$

Integrating (96) with respect to  $\tau$  over  $(0, t)$  gives,

$$(97) \quad \begin{aligned} & \int_0^t \|p_t(\tau)\|^2 d\tau + \|\nabla p(t)\|^2 + \frac{1}{2}|p(t)|_4^4 + \|p_0\|^2 \\ &= \|p(t)\|^2 + 2\langle f, p(t) \rangle + \|\nabla p_0\|^2 + \frac{1}{2}|p_0|_4^4 - 2\langle f, p_0 \rangle. \end{aligned}$$

Omitting the terms  $\|\nabla p(s)\|^2 + \frac{1}{2}|p(s)|_4^4 + \|p_0\|^2$  from the left hand side, controlling the right hand side with  $2\|p(s)\|^2 + 2\|f\|^2 + C\|\nabla p_0\|^2$ , and employing the uniform bound on  $\|p(s)\|^2$  given in equation (94) of corollary 5.23, we obtain (95).  $\square$

*Lemma 5.25.* Let  $p_0 \in H_0^1$  and  $f \in L^2$ . For all  $t \geq 0$  there holds,

$$(98) \quad \begin{aligned} \|\nabla p(t)\|^2 &\leq \|p(t)\|^2 + \|\nabla p(t)\|^2 \\ &\leq e^{-t/2}(\|p_0\|^2 + \|\nabla p_0\|^2) + \left(4\|f\|^2 + \frac{9}{4}\pi\right)(1 - e^{-t/2}). \end{aligned}$$

*Proof.* Multiplying the PDE in problem (P) by  $-2\Delta p(t)$  yields for almost all  $t \geq 0$ ,

$$\frac{d}{dt}\|\nabla p\|^2 + 2\|\Delta p\|^2 + 6\langle p^2\nabla p, \nabla p \rangle = 2\langle f + p, -\Delta p \rangle \leq \|f\|^2 + \|\Delta p\|^2 + 2\|\nabla p\|^2.$$

Omitting the positive term  $6\langle p^2\nabla p, \nabla p \rangle$  from the left hand side gives,

$$(99) \quad \frac{d}{dt}\|\nabla p\|^2 + \|\Delta p\|^2 \leq \|f\|^2 + 2\|\nabla p\|^2.$$

Adding (99) to (93) from the proof of lemma 5.22, yields

$$(100) \quad \frac{d}{dt}(\|p\|^2 + \|\nabla p\|^2) + \|\Delta p\|^2 \leq 2\|f\|^2 + \frac{9}{8}\pi.$$

Now employing the estimate

$$\frac{1}{2}(\|p\|^2 + \|\nabla p\|^2) \leq c\|\Delta p\|^2,$$

in (100), we obtain that for almost all  $t \geq 0$ ,

$$\frac{d}{dt}(\|p\|^2 + \|\nabla p\|^2) + \frac{1}{2}(\|p\|^2 + \|\nabla p\|^2) \leq 2\|f\|^2 + \frac{9}{8}\pi.$$

From this, (98) follows.  $\square$

*Corollary 5.26.* Let  $p_0 \in H_0^1$  and  $f \in L^2$ . It follows from (98) that for all  $t \geq 0$ ,

$$(101) \quad \|\nabla p(t)\|^2 \leq 2\|\nabla p_0\|^2 + 4\|f\|^2 + \frac{9}{4}\pi.$$

*Lemma 5.27.* For any  $p_0 \in H^2 \cap H_0^1$  and  $f \in H^1$ , there exists  $C > 0$ , depending on  $p_0$ ,  $f$ , and the imbedding  $H_0^1 \hookrightarrow L^\infty$ , such that for all  $t \geq 0$ ,

$$(102) \quad \|\Delta p(t)\|^2 \leq Ct.$$

*Proof.* We now formally multiply the PDE in problem (P) by  $2(-\Delta)^2 p(t)$  in  $L^2$  to obtain

$$(103) \quad \begin{aligned} \frac{d}{dt}\|\Delta p\|^2 + 2\|\nabla\Delta p\|^2 &= 2\langle \nabla f + \nabla p, \nabla\Delta p \rangle - 6\langle p^2\nabla p, \nabla\Delta p \rangle \\ &\leq 2\|\nabla f\|^2 + 2\|\nabla p\|^2 + 2\|\nabla\Delta p\|^2 + C\|\nabla p\|^2, \end{aligned}$$

where the constant  $C > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ . Omitting the term  $2\|\nabla\Delta p\|^2$  from (103) and integrating yields,

$$\|\Delta p(t)\|^2 \leq \|\Delta p_0\|^2 + \frac{1}{2}\|\nabla f\|^2 t + \int_0^t \frac{1}{2}\|\nabla p(\tau)\|^2 + C\|\nabla p(\tau)\|^2 d\tau,$$

and now with (101) of corollary 5.26, then we have equation (102) of the claim.  $\square$

*Lemma 5.28.* For any  $p_0 \in H^2 \cap H_0^1$ , and  $f \in H^1$ , there exists  $C > 0$ , depending on  $p_0$ ,  $f$ , and  $\Omega$ , such that for all  $t \geq 0$ ,

$$(104) \quad \int_0^t \|\nabla p_t(\tau)\|^2 d\tau \leq \|\nabla p_0\|^2 + Ct.$$

*Proof.* Multiply the PDE in problem (P) by  $-2\Delta p_t(t)$  to obtain

$$(105) \quad \begin{aligned} 2\|\nabla p_t\|^2 + \frac{d}{dt}\|\Delta p\|^2 &= 2\langle \nabla f + \nabla p, \nabla p_t \rangle - 6\langle p^2 \nabla p, \nabla p_t \rangle \\ &\leq 4\|\nabla f\|^2 + 4\|\nabla p\|^2 + \|\nabla p_t\|^2 + C\|\nabla p\|^2. \end{aligned}$$

Using (101), then (105) reduces to

$$(106) \quad \|\nabla p_t\|^2 + \frac{d}{dt}\|\nabla p\|^2 \leq C.$$

Integrating (106) with respect to  $t$  produces

$$(107) \quad \int_0^t \|\nabla p_t(\tau)\|^2 d\tau + \|\nabla p(t)\|^2 \leq \|\nabla p_0\|^2 + Ct.$$

Omitting the term  $\|\nabla p(t)\|^2$  from (107) above, we obtain (104) of the claim.  $\square$

*Lemma 5.29.* For any  $p_0 \in H^2 \cap H_0^1$ , and  $f \in H^1$ , there exist  $C_1, C_2 > 0$ , such that for all  $t \geq 0$ ,

$$(108) \quad \int_0^t \|p_{tt}(\tau)\|^2 d\tau \leq \|\nabla \mathcal{E}(p_0)\|^2 + C_1(\|\nabla \mathcal{E}(p_0)\|^2 + 1) + C_2.$$

Both  $C_1$  and  $C_2$  depend on  $p_0$  and the imbedding  $H_0^1 \hookrightarrow L^\infty$ . Additionally,  $C_2$  depends on  $f$ , and  $m(\Omega)$ .

*Proof.* Differentiating problem (P) with respect to  $t$  gives

$$(109) \quad \begin{cases} p_{tt} - \Delta p_t + 3p^2 p_t - p_t = 0 & \text{in } Q \\ p_t|_{\partial\Omega} = 0 \\ p_t(0, \cdot) = \mathcal{E}(h_0). \end{cases}$$

Formally multiplying the PDE in (109) by  $2p_{tt}(t)$  in  $L^2$  produces, for almost all  $t \geq 0$ ,

$$(110) \quad 2\|p_{tt}\|^2 + \frac{d}{dt}\|\nabla p_t\|^2 = -2\langle (3p^2 - 1)p_t, p_{tt} \rangle \leq 6\|p^2 p_t\|^2 + \|p_{tt}\|^2 + 2\|p_t\|^2.$$

Recall that

$$\|p^2 p_t\| \leq |p|_\infty^2 \|p_t\| \leq C\|p_t\|,$$

where  $C > 0$  depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ . Then, (110) becomes,

$$(111) \quad \|p_{tt}\|^2 + \frac{d}{dt}\|\nabla p_t\|^2 \leq C\|\nabla p_t\|^2 + 2\|p_t\|^2.$$

Omitting the first term  $\|p_{tt}(t)\|^2$  from (111), integrating with respect to  $t$ , and applying (95) of lemma 5.24, we now have

$$(112) \quad \begin{aligned} \|\nabla p_t(t)\|^2 &\leq \|\nabla p_t(0)\|^2 + C \int_0^t \|\nabla p_t(\tau)\|^2 d\tau + 2 \int_0^t \|p_t(\tau)\|^2 d\tau \\ &\leq \|\nabla p_t(0)\|^2 + C \int_0^t \|\nabla p_t(\tau)\|^2 d\tau + C', \end{aligned}$$

where  $C' > 0$  also depends on the imbedding  $H_0^1 \hookrightarrow L^\infty$ . With the aid of Gronwall's inequality, (112) becomes

$$\|\nabla p_t(t)\|^2 \leq (\|\nabla p_t(0)\|^2 + C') \exp \left( C \int_0^t \|\nabla p_t(\tau)\|^2 d\tau \right),$$

and with the aid of (101) of corollary 5.26,

$$(113) \quad \|\nabla p_t(t)\|^2 \leq (\|\nabla p_t(0)\|^2 + C') e^{C'' t},$$

where  $C'' > 0$  depends on  $h_0$ ,  $f$ , and  $\Omega$ .

Now integrating (111) with respect to  $t$  yields,

$$(114) \quad \int_0^t \|p_{tt}(\tau)\|^2 d\tau + \|\nabla p_t(t)\|^2 \leq \|\nabla p_t(0)\|^2 + C \int_0^t \|\nabla p_t(\tau)\|^2 d\tau + C'.$$

By omitting the second term  $\|\nabla p_t(t)\|^2$  of (114) and by employing (113) we arrive at (108) as claimed.  $\square$

With this at hand, we can now proceed to show that hypothesis 4.1 holds for the model problem.

**Proof of hypothesis 4.1.** Let  $B_3^\varepsilon$  be a bounded set in  $X_3^\varepsilon$ , and let  $(h_0, h_1) \in B_3^\varepsilon$ . Multiply the PDE in (91) by  $2z_t(t)$  in  $L^2$  to yield, for almost all  $t \geq 0$ ,

$$\begin{aligned} \frac{d}{dt} (\varepsilon \|z_t\|^2 + \|\nabla z\|^2) + 2\|z_t\|^2 &= -2\langle g(h) - g(p), z_t \rangle - 2\varepsilon \langle p_{tt}, z_t \rangle \\ &\leq L \|\nabla u - \nabla p\|^2 + 2\|z_t\|^2 + \varepsilon \|p_{tt}\|^2 \\ &\leq L \|\nabla z\|^2 + 2\|z_t\|^2 + \varepsilon \|z_t\|^2 + \varepsilon \|p_{tt}\|^2, \end{aligned}$$

where we employ the local Lipschitz continuity of the nonlinearity  $g$  on  $B_3^\varepsilon$ . Then,

$$\frac{d}{dt} [e^{-C_L t} (\varepsilon \|z_t\|^2 + \|\nabla z\|^2)] \leq \varepsilon e^{-C_L t} \|p_{tt}\|^2,$$

where  $C_L := \max\{L, 1\}$ . Integrating with respect to  $t$  in the compact interval  $[\tau_3, 2\tau_3]$ , where now

$$\tau_3^* := \max\{\tau_3^p, \tau_3^\varepsilon\},$$

and applying the result of lemma 5.29 leaves us with

$$\|(z(t), z_t(t))\|_{X_1^\varepsilon} \leq C\sqrt{\varepsilon}.$$

The constant  $C > 0$  depends on  $h_0$ ,  $f$ ,  $m(\Omega)$ , the imbedding  $H_0^1 \hookrightarrow L^\infty$ ,  $L > 0$ ,  $\tau_3^*$ , but not  $\varepsilon$ . This establishes hypothesis 4.1 with  $\gamma = \frac{1}{2}$ .  $\square$

We may now satisfy the remaining compatibility criteria by setting:

- (1) Recall that we already set  $N^* := \max\{N_{p*}, N_{\varepsilon*}\}$  above. Now we can define  $\mathfrak{I}_1 := \text{span}\{\omega_n(x)\}_{n=1}^{N^*}$  and  $\Sigma_1 := \mathfrak{I}_1 \times \mathfrak{I}_1$ .

- (2) Given  $\mathcal{W}^p \subset \mathfrak{I}_1$ , assume  $\mathcal{W}^\varepsilon = \mathcal{W}^p \times \mathcal{W}^p \subset \Sigma_1$ . It follows that  $\dim(\mathfrak{I}_1) = \dim(\Sigma_1) = N^*$ .
- (3) In the proof above we already set  $\tau_3^* = \max\{\tau_3^p, \tau_3^\varepsilon\}$ ; so replacing both  $\tau_3^p$  and  $\tau_3^\varepsilon$  with  $\tau_3^*$ , it also follows that  $\tau_3^p = \tau_3^\varepsilon$  and  $I_3^p = I_3^\varepsilon$ .
- (4) This is not needed here.

As of yet, we cannot apply theorem 4.7 because we cannot show that hypothesis 4.6 holds for the model problem. There is a difficulty in obtaining the estimates in hypothesis 4.6 because of a lack of information about the functions  $f \in \tilde{\mathcal{G}}_L^0$  and  $m_p \in \mathcal{G}_L^0$ . We will however show a different version of hypothesis 4.6 which yields a *weaker* robustness result.

*Hypothesis 5.30.* Assume that there exist  $C, C' > 0$ ,  $\lambda, \lambda' \in (0, 1]$  such that for all  $\eta > 0$  there exists  $N_0 \geq N^* + 1$ , depending on  $\eta$ , so that for any  $N \geq N_0$ ,

$$(115) \quad \sup_{\alpha \in \omega_{c_2^p}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_{Y_1} \leq C\eta^\lambda$$

and

$$(116) \quad \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3\beta\|_{Y_1} \leq C'\eta^{\lambda'}.$$

**Proof of hypothesis 5.30.** First we show (115). Fix

$$\alpha = (\chi, \psi) + m_\varepsilon(\chi, \psi) \in \omega_{c_2^p}^{K_\varepsilon}(C_2^\varepsilon)$$

and choose

$$\tilde{\beta} = \chi + m_p(\chi) \in \omega_{c_2^p}^{K_p}(C_2^0).$$

Then

$$\inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3\alpha - S_p(t)\beta\|_1 \leq \|S_p(t)\Pi_1^3\alpha - S_p(t)\tilde{\beta}\|_1.$$

The difference  $z(t) := S_p(t)\Pi_1^3\alpha - S_p(t)\tilde{\beta} = p_1(t) - p_2(t)$  solves

$$\begin{cases} z_t - \Delta z + g(p_1) - g(p_2) = 0 \\ z(t, 0) = z(t, \pi) = 0 \\ z(0) = \Pi_1^3\alpha - \tilde{\beta} = \Pi_1^3m_\varepsilon(\chi, \psi) - m_p(\chi). \end{cases}$$

Recall by lemma 4.4 that  $m_\varepsilon \in \mathcal{G}_L$  is such that  $m_\varepsilon(\chi, \psi) = (f(\chi), g(\psi))$  for some  $f \in \tilde{\mathcal{G}}_L^0$  and  $g \in \mathcal{G}_L^0$ ; thus,  $\Pi_1^3m_\varepsilon(\chi, \psi) = f(\chi)$ . Multiplying the above PDE in  $L^2$  by  $-2\Delta z(t)$  yields

$$\frac{d}{dt} \|\nabla z\|^2 + 2\|\Delta z\|^2 \leq C_L \|\nabla z\|^2 + 2\|\Delta z\|^2.$$

Hence, for all  $t \in I_3^p := [\tau_3^p, 2\tau_3^p]$ ,

$$\|z(t)\|_1^2 \leq e^{2C_L\tau_3^p} \|z(0)\|_1^2;$$

thus, for all  $t \in I_3^p$ ,

$$(117) \quad \|S_p(t)\Pi_1^3\alpha - S_p(t)\tilde{\beta}\|_1 \leq C(\|f(\chi)\|_1 + \|m_p(\chi)\|_1).$$

Recall  $m_p(\chi) \in \mathfrak{I}_2$  and  $f(\chi) \in \tilde{\mathfrak{I}}_2$ . We now show that there is a function  $s(N^*)$  such that  $\|m_p(\chi)\|_1 \leq s(N^*)$  and  $s(N^*) \rightarrow 0$  as  $N^* \rightarrow \infty$ . Since the “tail” of the

Fourier series for  $m_p(\chi)$  converges in  $L^2$ ,

$$(118) \quad \|m_p(\chi)\|_1 = \left\| \sum_{n=N^*+1}^{\infty} \langle m_p(\chi), w_n \rangle w_n \right\|_1 \leq \sum_{n=N^*+1}^{\infty} |\langle m_p(\chi), w_n \rangle| \|w_n\|_1.$$

Using the normalized eigenfunctions  $\{\sqrt{\frac{2}{\pi}} \sin(nx)\}$  of the Laplacian  $-\Delta$  in  $L^2$  with domain  $H^2 \cap H_0^1$  and the corresponding eigenvalues  $\lambda_n = n^2$ , then

$$\|w_n\|_1^2 = \|\nabla w_n\|^2 = \langle w_n, -\Delta w_n \rangle = \lambda_n \|w_n\|^2 = n^2,$$

and

$$\begin{aligned} \langle m_p(\chi), w_n \rangle &= \frac{1}{\lambda_n} \langle m_p(\chi), -\Delta w_n \rangle = \frac{1}{\lambda_n} \langle -\Delta m_p(\chi), w_n \rangle \\ &= \frac{1}{\lambda_n^2} \langle -\Delta m_p(\chi), -\Delta w_n \rangle = \frac{1}{\lambda_n^2} \langle \nabla \Delta m_p(\chi), \nabla w_n \rangle \\ &\leq \frac{1}{\lambda_n^2} \|m_p(\chi)\|_3 \|w_n\|_1 \leq \frac{n}{n^4} \|m_p(\chi)\|_3 = n^{-3} \|m_p(\chi)\|_3. \end{aligned}$$

Thus, from (118)

$$\|m_p(\chi)\|_1 \leq \sum_{n=N^*+1}^{\infty} |\langle m_p(\chi), w_n \rangle| \|w_n\|_1 \leq \|m_p(\chi)\|_3 \sum_{n=N^*+1}^{\infty} \frac{1}{n^2}.$$

A similar result holds for  $f(\chi) \in \tilde{\mathcal{I}}_2$ . The Fourier series converges in  $H_0^1$ , so

$$(119) \quad \|f(\chi)\|_1 \leq \sum_{n=N^*+1}^{\infty} |\langle f(\chi), w_n \rangle_1| \|w_n\|_1.$$

Here,

$$\begin{aligned} \langle f(\chi), w_n \rangle_1 &= \langle \nabla f(\chi), \nabla w_n \rangle = \langle -\Delta f(\chi), w_n \rangle \\ &= \frac{1}{\lambda_n^2} \langle -\Delta f(\chi), -\Delta w_n \rangle = \frac{1}{\lambda_n^2} \langle \nabla \Delta f(\chi), \nabla w_n \rangle \\ &\leq \frac{1}{\lambda_n^2} \|f(\chi)\|_3 \|w_n\|_1 \leq \frac{n}{n^4} \|f(\chi)\|_3 \\ &= n^{-3} \|f(\chi)\|_3. \end{aligned}$$

Now (119) becomes

$$\|f(\chi)\|_1 \leq \sum_{n=N^*+1}^{\infty} |\langle f(\chi), w_n \rangle_1| \|w_n\|_1 \leq \|f(\chi)\|_3 \sum_{n=N^*+1}^{\infty} \frac{1}{n^2}.$$

Hence, equation (117) becomes

$$\begin{aligned} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\tilde{\beta}\|_1 &\leq C(\|f(\chi)\|_3 + \|m_p(\chi)\|_3) \\ &\leq C \sum_{n=N^*+1}^{\infty} \frac{1}{n^2}, \end{aligned}$$

for which, given any  $\eta > 0$ , there exists  $N_0 \geq N^* + 1$ , sufficiently large and depending on  $\eta$ , so that

$$(120) \quad \|S_p(t)\Pi_1^3 \alpha - S_p(t)\tilde{\beta}\|_1 \leq C\eta^{1/2}.$$

Since  $\Pi_1^3 \alpha = \chi + f(\chi)$  is in the bounded set  $\Pi_1^3 \omega_{c_*}^{K_\varepsilon}(C_2^\varepsilon)$  in  $Y_3$ , then such  $N_0$  is uniform in  $(\chi, \psi)$ . Therefore, for each  $N \geq N_0$ , the model problem satisfies (115) with  $\lambda = \frac{1}{2}$ .

Now to show (116) holds.  $\square$

We now describe the weaker robustness result for the generic case. Since hypothesis 5.30 holds for the model problem, the following result (theorem 5.31 below) holds for the application in this section. Following the first part of the proof of theorem 4.7, we arrive at inequality (36):

$$\partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq C_3 \varepsilon^{\gamma_0} + \sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1}.$$

where  $C_3 > 0$  and  $\gamma_0 \in (0, \frac{1}{2}]$  are both independent of  $\varepsilon$ . Now by hypothesis 5.30 (instead of hypothesis 4.6) there exist  $C_4 > 0$ ,  $\lambda \in (0, 1]$  such that for all  $\eta > 0$  there exists  $N_0 \geq N^* + 1$ , depending on  $\eta$ , so that for any  $N \geq N_0$ ,

$$\sup_{\alpha \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \inf_{\beta \in \omega_{c_2^p}^{K_p}(C_2^0)} \|S_p(t)\Pi_1^3 \alpha - S_p(t)\beta\|_{Y_1} \leq C_4 \eta^\lambda.$$

Thus,

$$\partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq C_3 \varepsilon^{\gamma_0} + C_4 \eta^\lambda.$$

By choosing  $\eta = \varepsilon$ , and setting  $\Lambda_1 := 2 \max\{C_3, C_4\}$  and  $\phi_1 := \min\{\gamma_0, \lambda\}$ , then there is  $N_0 \geq N^* + 1$  so that for any  $N \geq N_0$ , the  $N$ -dimensional compact inertial manifolds  $\widehat{\mathcal{M}}_3^\varepsilon$  and  $\widehat{\mathcal{M}}_3^0$  satisfy

$$(121) \quad \partial_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq \Lambda_1 \varepsilon^{\phi_1}.$$

On the other hand, following the second part of the proof of theorem 4.7, we have the following inequality before the use of hypothesis 4.6:

$$\partial_{X_1^\varepsilon}(\mathcal{L}\widehat{\mathcal{M}}_3^0, \widehat{\mathcal{M}}_3^\varepsilon) \leq C'_3 \varepsilon^{\gamma'_0} + \sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1}.$$

where  $C'_3 > 0$  and  $\gamma'_0 \in (0, \frac{1}{2}]$  are both independent of  $\varepsilon$ . Now with hypothesis 5.30 there exist  $C'_4 > 0$ ,  $\lambda' \in (0, 1]$  such that for all  $\eta > 0$  there exists  $N_0 \geq N^* + 1$ , depending on  $\eta$ , so that for any  $N \geq N_0$ ,

$$\sup_{\alpha \in \omega_{c_2^p}^{K_p}(C_2^0)} \inf_{\beta \in \omega_{c_2^\varepsilon}^{K_\varepsilon}(C_2^\varepsilon)} \|S_p(t)\alpha - S_p(t)\Pi_1^3 \beta\|_{Y_1} \leq C_4 \eta^{\lambda'}.$$

Thus,

$$\partial_{X_1^\varepsilon}(\mathcal{L}\widehat{\mathcal{M}}_3^0, \widehat{\mathcal{M}}_3^\varepsilon) \leq C_3 \varepsilon^{\gamma_0} + C'_4 \eta^{\lambda'}.$$

By choosing  $\eta = \varepsilon$ , and setting  $\Lambda_2 := 2 \max\{C'_3, C'_4\}$  and  $\phi_2 := \min\{\gamma'_0, \lambda'\}$ , then there is  $N_0 \geq N^* + 1$  so that for any  $N \geq N_0$ , the  $N$ -dimensional compact inertial manifolds  $\widehat{\mathcal{M}}_3^\varepsilon$  and  $\widehat{\mathcal{M}}_3^0$  satisfy

$$(122) \quad \partial_{X_1^\varepsilon}(\mathcal{L}\widehat{\mathcal{M}}_3^0, \widehat{\mathcal{M}}_3^\varepsilon) \leq \Lambda_2 \varepsilon^{\phi_2}.$$

Combining (121) and (122) yields

$$\text{dist}_{X_1^\varepsilon}(\widehat{\mathcal{M}}_3^\varepsilon, \mathcal{L}\widehat{\mathcal{M}}_3^0) \leq \Lambda \varepsilon^\phi$$

where  $\Lambda := \max\{\Lambda_1, \Lambda_2\}$  and  $\phi := \min\{\phi_1, \phi_2\}$ , and therefore the family  $(\mathbb{M}_3^\varepsilon)_{\varepsilon \in [0, 1]}$  is robust at  $\varepsilon_0 = 0$  in the topology of  $X_1^\varepsilon$ .

Through hypothesis 5.30, the weak robustness result means that by increasing the dimension of the underlying manifolds, we can make the corresponding compact inertial manifolds sufficiently “close.” Moreover, for all manifolds whose dimension exceeds the prescribed number  $N_0$ , the corresponding compact inertial manifolds are also sufficiently “close.” We have shown the following.

*Theorem 5.31.* Let  $\widehat{\mathcal{M}}_3^0$  be the compact inertial manifold constructed above, and for each  $\varepsilon \in (0, 1]$ , let  $\widehat{\mathcal{M}}_3^\varepsilon$  be the compact inertial manifold constructed above while under the compatibility criteria less #4. If hypotheses 4.1, 4.5, and 5.30 hold, then the family  $(\mathbb{M}_3^\varepsilon)_{\varepsilon \in [0, 1]}$  defined in (32) satisfies the following: there exist  $\Lambda > 0$  and  $\phi \in (0, 1]$  such that for every  $\varepsilon \in (0, 1]$  there exists  $N_0 \geq N^* + 1$  in which for every  $N \geq N_0$ ,

$$\text{dist}(\mathbb{M}_3^\varepsilon, \mathbb{M}_3^0) \leq \Lambda \varepsilon^\phi.$$

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